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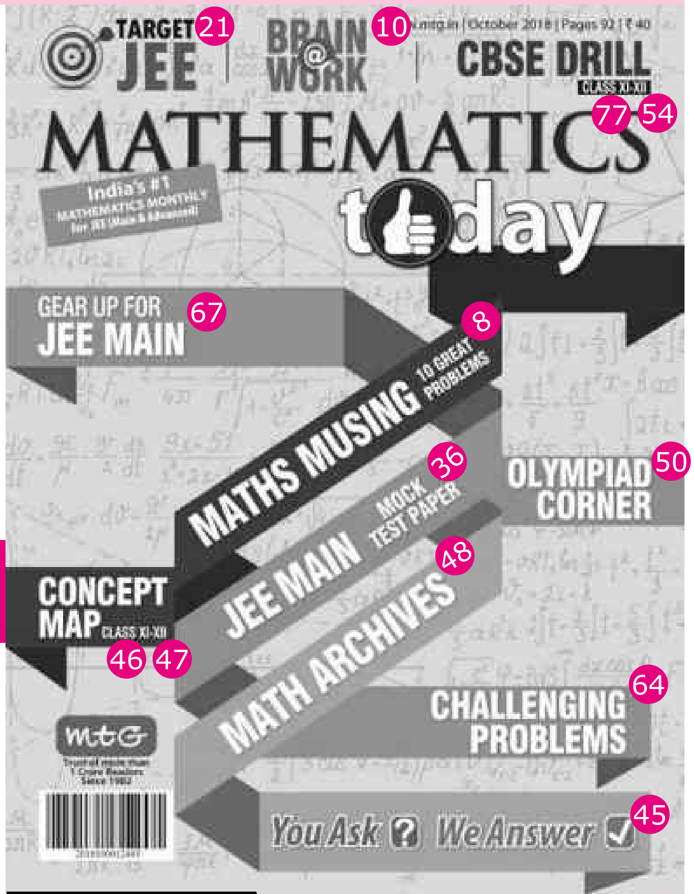

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 190

JEE MAIN

1. If ABC is a triangle and $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P., then the minimum value of $\cot\frac{B}{2} =$

(a) $\sqrt{3}$ (b) 1 (c) $1/\sqrt{2}$ (d) $1/\sqrt{3}$

2. If a is a root of the equation $x^2 - 3x + 1 = 0$, then the value of $\frac{a^3}{a^6+1}$ is

(a) 1/14 (b) 1/15 (c) 1/16 (d) 1/18

3. Coefficient of $x^{(n^2+n-14)/2}$ in $(x-1)(x^2-2)(x^3-3)(x^4-4)\ldots(x^n-n)$, $(n \ge 8)$ is

(a) 13 (b) 21
(c) 28 (d) none of these

4. If $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$ $= k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then k is equal to

(a) 1 (b) -1 (c) $\alpha\beta$ (d) $1/\alpha\beta$

5. Consider the integrals

$$I_1 = \int_0^1 e^{-x}\cos^2 x\,dx,\quad I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx,$$

$$I_3 = \int_0^1 e^{-x^2}dx \text{ and } I_4 = \int_0^1 e^{-(1/2)x^2}dx.$$ The greatest of these integrals is

(a) I_1 (b) I_2 (c) I_3 (d) I_4

JEE ADVANCED

6. If $A = \begin{pmatrix} 5 & -3 \\ 111 & 336 \end{pmatrix}$ and $\det(-3A^{2013} + A^{2014}) = \alpha^\alpha\beta^2 \times (1+\gamma+\gamma^2)$, then

(a) $\alpha = 2013$ (b) $\beta = 3$
(c) $\gamma = 10$ (d) None of these

COMPREHENSION

Let us consider the equation

$$\frac{\cos^4 x}{a} + \frac{\sin^4 x}{b} = \frac{1}{a+b}, x \in \left[0, \frac{\pi}{2}\right];\ a, b > 0$$

7. In the given equation

(a) $\frac{\sin^4 x}{b} = \frac{\cos^4 x}{a}$ (b) $\frac{\sin x}{a} = \frac{\cos x}{b}$

(c) $\frac{\sin^4 x}{b^2} = \frac{\cos^4 x}{a^2}$ (d) $\frac{\sin^2 x}{a} = \frac{\cos^2 x}{b}$

8. The value of $\sin^2 x$ in terms of a and b is

(a) $\sqrt{ab}$ (b) $\frac{b}{a+b}$ (c) $\frac{b^2-a^2}{a^2+b^2}$ (d) $\frac{a^2+b^2}{b^2-a^2}$

INTEGER TYPE

9. $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $p(1) = 10$, $p(2) = 20$, $p(3) = 30$, then value of $\frac{p(12)+p(-8)}{10} - 1980$ is

MATRIX MATCH

10. Match the following:

	Column-I		Column-II
P.	$\vec a, \vec b, \vec c$ are non-coplanar vectors, then $(\vec a+\vec b+\vec c)\cdot[(\vec a+\vec b)\times(\vec a+\vec c)] =$	1.	$\frac{1}{4}a^2b^2$
Q.	If $\vec b, \vec c$ are any two non-collinear unit vectors and $\vec a$ is any vector, then $(\vec a\cdot\vec b)\vec b + (\vec a\cdot\vec c)\vec c + \frac{\vec a\cdot(\vec b\times\vec c)}{\lvert\vec b\times\vec c\rvert}(\vec b\times\vec c) =$	2.	$-[\vec a\ \vec b\ \vec c]$
R.	If $\vec a, \vec b, \vec c$ are non-collinear vectors, then $[\vec a+\vec b+\vec c, \vec a-\vec c, \vec a-\vec b] =$	3.	$\vec a$
		4.	$-3[\vec a\vec b\vec c]$

	P	Q	R
(a)	3	2	1
(b)	2	3	4
(c)	1	2	3
(d)	1	4	4

BRAIN @ WORK

Application of Derivatives

TANGENTS AND NORMALS

- Let $y = f(x)$ be the given curve. The equation of the tangent at (x_1, y_1) is

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1,\ y_1)}(x - x_1)$$

or $\quad y - f(x_1) = \left(\frac{dy}{dx}\right)_{(x_1,\ y_1)}(x - x_1)$

- The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1),$$

provided that $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \neq 0$

ANGLE OF INTERSECTION OF TWO CURVES

- Let $y = f(x)$ and $y = g(x)$ be two given intersecting curves. Angle of intersection of these curves is defined as the acute angle between the tangents that can be drawn to the given curves at the point of intersection.

i.e., $\tan\theta = \left|\frac{m_1 + m_2}{1 - m_1 m_2}\right|$, where m_1 and m_2 are the respective slopes of the two curves.

MONOTONICITY

Let $y = f(x)$ be a given function with 'D' as its domain. Let $D_1 \subseteq D$, then

- **Increasing Function :** $f(x)$ is said to be increasing in D_1 if for every $x_1, x_2 \in D_1, x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$. $f(x)$ is increasing in $[a, b]$ if $f'(x) > 0 \ \forall \ x \in (a, b)$.
- **Non-decreasing function:** $f(x)$ is said to be non-decreasing in D_1 if for every $x_1, x_2 \in D_1, x_1 > x_2$ $\Rightarrow f(x_1) \geq f(x_2)$. $f(x)$ is non-decreasing in $[a, b]$ if $f'(x) \geq 0 \ \forall \ x \in (a, b)$.
- **Decreasing function:** $f(x)$ is said to be decreasing in D_1 if for every $x_1, x_2 \in D_1, x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$. $f(x)$ is decreasing in $[a, b]$ if $f'(x) < 0 \ \forall \ x \in (a, b)$.
- **Non-increasing function :** $f(x)$ is said to be non-increasing in D_1 if for every $x_1, x_2 \in D_1, x_1 > x_2$ $\Rightarrow f(x_1) \leq f(x_2)$. $f(x)$ is non-increasing in $[a, b]$ if $f'(x) \leq 0 \ \forall \ x \in (a, b)$.

Notes :

(i) If $f'(x) \geq 0 \ \forall \ x \in (a, b)$ and points which make $f'(x)$ equal to zero (in between (a, b)) don't form an interval, then $f(x)$ would be increasing in $[a, b]$.

(ii) If $f'(x) \leq 0 \ \forall \ x \in (a, b)$ and points which make $f'(x)$ equal to zero (in between (a, b)) don't form an interval, then $f(x)$ would be decreasing in $[a, b]$.

(iii) If $f(0) = 0$ and $f'(x) \geq 0 \ \forall \ x \in R$, then $f(x) \leq 0 \ \forall \ x \in (-\infty, 0)$ and $f(x) \geq 0 \ \forall \ x \in (0, \infty)$

(iv) If $f(0) = 0$ and $f'(x) \leq 0 \ \forall \ x \in R$, then $f(x) \geq 0 \ \forall \ x \in (-\infty, 0)$ and $f(x) \leq 0 \ \forall \ x \in (0, \infty)$

(v) A function is said to be monotonic if it is either increasing or decreasing.

(vi) The points for which $f'(x)$ is equal to zero or doesn't exist are called critical points. Here it should also be noted that critical points are the interior points of an interval.

(vii) The stationary points are the points where $f'(x) = 0$ in the domain.

MAXIMA AND MINIMA

- **Local maxima and Local minima:** Let $y = f(x)$ be a function defined at $x = a$ and also in the vicinity of the point $x = a$. Then, $f(x)$ is said to have a local maximum at $x = a$, if the value of the function at $x = a$ is greater than the value of the function at the neighbouring points of $x = a$.

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Similarly $f(x)$ is said to have a local minimum at $x = a$, if the value of the function at $x = a$ is less than the value of the function at the neighbouring points of $x = a$.

TEST FOR LOCAL MAXIMUM/ MINIMUM

- **Test for local maximum/minimum at $x = a$ if $f(x)$ is differentiable at $x = a$:**
- Let $f(x)$ is differentiable at $x = a$ and it is a critical point of the function, if $f'(a) = 0$ and if $f'(x)$ changes its sign while passing through the point $x = a$, then
  (a) $f(x)$ would have a local maximum at $x = a$ if $f'(a - 0) > 0$ and $f'(a + 0) < 0$. It means that $f'(x)$ should change its sign from positive to negative.
  (b) $f(x)$ would have a local minimum at $x = a$ if $f'(a - 0) < 0$ and $f'(a + 0) > 0$. It means that $f'(x)$ should change its sign from negative to positive.
  (c) If $f(x)$ doesn't change its sign while passing through $x = a$, then $f(x)$ would have neither a maximum nor minimum at $x = a$.
- **Second order derivative test for maxima and minima**
  Let $f(x)$ be a differentiable function on an interval I. Let $a \in I$ and $f''(x)$ is continuous at $x = a$. Then
  (a) $x = a$ is a point of local maximum if $f'(a) = 0$ and $f''(a) < 0$.
  (b) $x = a$ is a point of local minimum if $f'(a) = 0$ and $f''(a) > 0$.
  (c) If $f'(a) = f''(a) = 0$ and $f'''(a) \neq 0$ if exists, then $x = a$ is neither a point of local maximum nor a point of local minimum.
- **Test for local maximum/minimum at $x = a$ if $f(x)$ is not differentiable at $x = a$:**
  Case 1: When $f(x)$ is continuous at $x = a$ and $f'(a - h)$ and $f'(a + h)$ exist and are non-zero, then $f(x)$ has a local maximum or minimum at $x = a$ if $f'(a - h)$ and $f'(a + h)$ are of opposite signs.
- If $f'(a - h) > 0$ and $f'(a + h) < 0$, then $x = a$ will be a point of local maximum.
  If $f'(a - h) < 0$ and $f'(a + h) > 0$, then $x = a$ will be a point of local minimum.
  Case 2: When $f(x)$ is continuous and $f'(a - h)$ and $f'(a + h)$ exist but one of them is zero, we should compare the information about the existence of local maximum / minimum from the basic definition of local maximum / minimum.
  Case 3: If $f(x)$ is not continuous at $x = a$ and $f'(a - h)$ and/or $f'(a + h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x = a$.

CONCEPT OF GLOBAL MAXIMUM/MINIMUM

Let $y = f(x)$ be a given function with domain D. Let $[a, b] \subseteq D$. Global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$. Global maximum and minimum in $[a, b]$ would always occur at critical points of $f(x)$ within $[a, b]$ or at the end points of the interval.

- **Rolle's Theorem :** It states that if $y = f(x)$ be a given function and satisfies following conditions :
  - $f(x)$ be continuous in $[a, b]$
  - $f(x)$ be differentiable in (a, b)
  - $f(a) = f(b)$,

  then $f'(c) = 0$ at least once for some $c \in (a, b)$.
- **Lagrange's Mean Value Theorem:** It states that if $y = f(x)$ be a given function and satisfies the following conditions :
  - $f(x)$ be continuous in $[a, b]$
  - $f(x)$ be differentiable in (a, b)

  then, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ at least once for some $c \in (a, b)$.

PROBLEMS

1. The tangent at $(t, t^2 - t^3)$ on the curve $y = x^2 - x^3$ meets the curve again at Q, then abscissa of Q must be
(a) $1 + 2t$ (b) $1 - 2t$ (c) $-1 - 2t$ (d) $2t - 1$

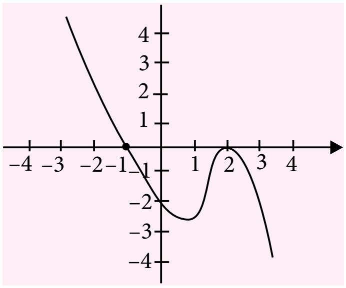
2. The point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is meet by the curve $xy = 1 - y$, is given by
(a) $(0, -1)$ (b) $(1, 1)$
(c) $(0, 1)$ (d) none of these

3. The tangent at any point of the curve $x^3 + y^3 = 2$ cuts off length p, q on the coordinate axes, the value of $p^{-3/2} + q^{-3/2} =$
(a) $2^{-3/2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\sqrt{2}$ (d) None of these

4. The number of points in the rectangle $\{(x, y) | -12 \le x \le 12, -3 \le y \le 3\}$ which lie on the curve $y = x + \sin x$ and at which the tangent to the curve is parallel to the x-axis is
(a) 0 (b) 2 (c) 4 (d) 8

5. A curve with equation of the form $y = ax^4 + bx^3 + cx + d$ has zero gradient at the point $(0, 1)$ and also touches the x-axis at the point $(-1, 0)$, then the value of x for which the curve has a negative gradient are
(a) $x > -1$ (b) $x < 1$
(c) $x < -1$ (d) $-1 \le x \le 1$

6. A cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ has a graph which touches the x-axis at 2, has another x-intercept at –1 and has y-intercept at –2 as shown. The value of, $a + b + c + d$ equals to



(a) –2 (b) –1 (c) 0 (d) 1

7. A horse runs along a circle with a speed of 20 km/hr. A lantern is at the centre of the circle. A fence is along the tangent to the circle at the point at which the horse starts. The speed (in km/hr) with which the shadow of the horse move along the fence at the moment when it covers 1/8 of the circle is

(a) 20 (b) 60 (c) 30 (d) 40

8. The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2}+\frac{b}{y^2}=1$ is proportional to

(a) square of the abscissa of the point of tangency
(b) square root of the abscissa of the point of tangency
(c) cube of the abscissa of the point of tangency
(d) cube root of the abscissa of the point of tangency

9. At any two points of the curve represented parametrically by $x = a(2\cos t - \cos 2t)$; $y = a(2\sin t - \sin 2t)$ the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by

(a) $2\pi/3$ (b) $3\pi/4$ (c) $\pi/2$ (d) $\pi/3$

10. Consider the function $f(x)=\begin{cases} 2+x^3, & \text{if } x \le 1 \\ 3x, & \text{if } x > 1 \end{cases}$, then

(a) f is continuous on [–1, 2] but is not differentiable on (–1, 2)
(b) mean value theorem is not applicable for the function on [–1, 2]
(c) mean value theorem is applicable on [–1, 2] and the value of $c = 1$
(d) mean value theorem is applicable on [–1, 2] and the value of c is $\pm\frac{\sqrt{5}}{3}$

11. Let $f(x) = 1 + x^m(x - 1)^n$ where $m, n \in N$. Then in (0, 1) the equation $f'(x) = 0$ has

(a) no root (b) at least one root
(c) at most one root (d) exactly one root

12. If $f(x)=\left(\frac{\sqrt{3a-5}}{1-a}-1\right)x^5 - 3x + \log 3 \ \forall\ x \in R$ is decreasing function then value of a lies in the interval

(a) $\left[\frac{3}{5},\infty\right)$ (b) $(-\infty, 1)$
(c) $\left[\frac{5}{3},\infty\right)$ (d) $(1, \infty)$

13. Function $f(x)=\frac{\lambda\sin x+3\cos x}{2\sin x+6\cos x}$ is monotonically increasing when

(a) $\lambda < 1$ (b) $\lambda > 1$ (c) $\lambda < 2$ (d) $\lambda > 2$

14. Given that f is a real valued differentiable function such that $f(x) \cdot f'(x) < 0$, for all real x it follows that

(a) $f^2(x)$ is increasing function
(b) $f^2(x)$ is decreasing function
(c) $f(x)$ is increasing function
(d) $f(x)$ is decreasing function

15. A function $y = f(x)$ is given by $x=\frac{1}{1+t^2}$ and $y=\frac{1}{t(1+t^2)}$ for all $t > 0$, then f is

(a) increasing in (0, 3/2) and decreasing in (3/2, ∞)
(b) increasing in (0, 1)
(c) increasing in (0, ∞)
(d) decreasing in (0, 1)

16. Number of roots of the function $f(x)=\frac{1}{(x+1)^3}-3x+\sin x$ is

(a) 0 (b) 1
(c) 2 (d) more than 2

17. Let $f : [-1, 2] \to R$ be differentiable such that $0 \le f'(t) \le 1$ for $t \in [-1, 0]$ and $-1 \le f'(t) \le 0$ for $t \in [0, 2]$. Then

(a) $-2 \le f(2) - f(-1) \le 1$ (b) $1 \le f(2) - f(-1) \le 2$
(c) $-3 \le f(2) - f(-1) \le 0$ (d) $-2 \le f(2) - f(-1) \le 0$

18. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall\ x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in

(a) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ (b) $\left(0,\frac{\pi}{4}\right)$
(c) $\left(0,\frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{6},\frac{\pi}{2}\right)$

19. Let $f(x)=\begin{cases}|x^3+x^2+3x+\sin x|\left(3+\sin\dfrac{1}{x}\right), & x\neq 0\\ 0 & , x=0\end{cases}$

Then value of point where $f(x)$ attains its minimum is

(a) 0 (b) 1 (c) 3 (d) infinite

20. The minimum value of the function

$$f(x)=\frac{x^p}{p}+\frac{x^{-q}}{q}, \text{ where } \frac{1}{p}+\frac{1}{q}=1, p>1 \text{ is}$$

(a) 1 (b) 0

(c) 2 (d) None of these

21. If $a < b < c < d$ and $x \in R$, then the least value of the function $f(x) = |x - a| + |x - b| + |x - c| + |x - d|$ is

(a) $c - d + b - a$ (b) $c + d - b - a$

(c) $c + d - b + a$ (d) $c - d + b + a$

22. The set of values of p for which the points of extremum of the function $f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$ lie in the interval $(-2, 4)$, is

(a) $(-3, 5)$ (b) $(-3, 3)$ (c) $(-1, 3)$ (d) $(-1, 5)$

23. Number of solution(s) satisfying the equation, $3x^2 - 2x^3 = \log_2(x^2 + 1) - \log_2 x$ is

(a) 1 (b) 2

(c) 3 (d) none of these

24. The equation of the line through (3, 4) which cuts the first quadrant in a triangle of minimum area, is

(a) $4x + 3y - 24 = 0$ (b) $3x + 4y - 12 = 0$

(c) $2x + 4y - 12 = 0$ (d) $3x + 2y - 24 = 0$

25. The minimum value of $a\tan^2\theta + b\cot^2\theta$ equals the maximum value of $a\sin^2\theta + b\cos^2\theta$ where $a > b > 0$, when

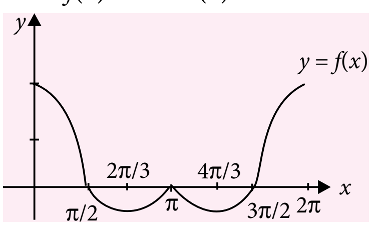
(a) $a = b$ (b) $a = 2b$ (c) $a = 3b$ (d) $a = 4b$

26. Let h be a twice continuously differentiable positive function on an open interval J. Let $g(x) = \ln(h(x))$ for each $x \in J$. Suppose $(h'(x))^2 > h''(x)\,h(x)$ for each $x \in J$. Then

(a) g is increasing on J (b) g is decreasing on J

(c) g is concave up on J (d) g is concave down on J

27. The graph of $y = f(x)$ is shown. Let $F(x)$ be an antiderivative of $f(x)$. Then $F(x)$ has



(a) points of inflexion at $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ and 2π, a local maximum at $x = \pi/2$, and a local minimum at $x = 3\pi/2$

(b) points of inflexion at $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ and 2π, a local minimum at $x = \pi/2$ and a local maximum at $x = 3\pi/2$

(c) point of inflexion at $x = \pi$, a local maximum at $x = \pi/2$, and a local minimum at $x = 3\pi/2$

(d) point of inflexion at $x = \pi$, a local minimum at $x = \pi/2$, and a local maximum at $x = 3\pi/2$

28. The value of the real number a having the property $f(a) = a$, is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$, is

(a) 1 (b) 2 (c) 3 (d) -1

29. For $a \in [\pi, 2\pi]$ and $n \in Z$, the critical points of $f(x) = \frac{1}{3}\sin a\tan^3 x + (\sin a - 1)\tan x + \sqrt{\frac{a-2}{8-a}}$ are

(a) $x = n\pi$ (b) $x = 2n\pi$

(c) $x = (2n + 1)\pi$ (d) none of these

30. For $0 < a \le 1$ and $b \in R$, then in $(-a, a)$ the function, $f(x) = ax^3 - 3ax + b$

(a) has exactly 2 roots (b) can not have a root

(c) has atmost one root (d) has more than two roots

31. The coordinates of the points on the curve, $5x^2 - 6xy + 5y^2 = 4$ which are the nearest to the origin are

(a) $\left(0,\frac{2}{\sqrt{5}}\right),\left(0,\frac{-2}{\sqrt{5}}\right)$ (b) $\left(\frac{1}{2},\frac{-1}{2}\right),\left(\frac{-1}{2},\frac{1}{2}\right)$

(c) $\left(\frac{2}{\sqrt{5}},0\right),\left(-\frac{2}{\sqrt{5}},0\right)$ (d) None of these

32. If $f(x)=\int_{x^2}^{x^3}\frac{dt}{\ln t}, x>0, \neq 1$ then

(a) $f(x)$ is an increasing function

(b) $f(x)$ has a minima at $x = 1$

(c) $f(x)$ is a decreasing function

(d) $f(x)$ has a maxima at $x = 1$.

33. $f : [1, \infty) \to R$ such that $f(x)$ is a monotonic and differentiable function and $f(1) = 1$, then number of solutions of the equation $f(f(x)) = \dfrac{1}{x^2-2x+2}$ is/are

(a) 2 (b) 1 (c) infinite (d) zero

34. The function $f(x)=\dfrac{x^2-3x+2}{x^2+2x-3}$ is

(a) max. at $x = -3$
(b) min. at $x = -3$ and max. at $x = 1$
(c) increasing in its domain
(d) none of these

35. Let N be any four digit number say $x_1\,x_2\,x_3\,x_4$. Then maximum value of $\dfrac{N}{x_1+x_2+x_3+x_4}$ is equal to

(a) 1000 (b) $\dfrac{1111}{4}$
(c) 800 (d) none of these

36. If the equation $x^5 - 10a^3x^2 + b^4x + c^5 = 0$ has three equal roots, then

(a) $2b^2 - 10a^3b^2 + c^5 = 0$ (b) $6a^5 + c^5 = 0$
(c) $2c^5 - 10a^3b^2 + b^4c^5 = 0$ (d) $b^4 = 15a^5$

37. If a, b are real numbers such that $x^3 - ax^2 + bx - 6 = 0$ has its roots real and positive, then minimum value of b is

(a) 1 (b) 2
(c) $3(36)^{1/3}$ (d) none of these

38. If $f''(x) + f'(x) + f^2(x) = x^2$ be the differential equation of a curve and let P be the point of maximum, then number of normals which can be drawn from P to $x^2 - y^2 = a^2$ is / are

(a) 2 (b) 1
(c) 0 (d) either 1 or 2

39. A function $g(\theta)=\int_0^{\sin^2\theta} f(x)dx+\int_0^{\cos^2\theta} f(x)dx$ is defined in the interval $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, where $f(x)$ is an increasing function, then $g(\theta)$ is increasing in the interval

(a) $\left(-\dfrac{\pi}{2},0\right)$ (b) $\left(-\dfrac{\pi}{2},-\dfrac{\pi}{4}\right)$
(c) $\left(0,\dfrac{\pi}{4}\right)$ (d) $\left(-\dfrac{\pi}{4},0\right)$

40. The minimum value of

$$(x_1-x_2)^2+\left(\frac{x_1^2}{20}-\sqrt{(17-x_2)(x_2-13)}\right)^2$$

where $x_1 \in R^+$ and $x_2 \in (13, 17)$, is

(a) $(5\sqrt{2}-2)^2$ (b) $(5\sqrt{2}-2)$
(c) $(5\sqrt{2}+2)^2$ (d) none of these

SOLUTIONS

1. (b) : The given curve is $y = x^2 - x^3$

Slope of tangent, $\left.\dfrac{dy}{dx}\right|_P = 2x - 3x^2 = 2t - 3t^2$

Equation of tangent is

$y - (t^2 - t^3) = (2t - 3t^2)(x - t)$

$\Rightarrow y - (t^2 - t^3) = (2t - 3t^2)\,x - (2t - 3t^2)t$

$\Rightarrow y = (2t - 3t^2)\,x - 2t^2 + 3t^3 + t^2 - t^3$

$\Rightarrow y = (2t - 3t^2)\,x + 2t^3 - t^2$ ...(i)

Solving (i) and $y = x^2 - x^3$, we get

$(2t - 3t^2)\,x + 2t^3 - t^2 = x^2 - x^3$

$\Rightarrow x^3 - x^2 + (2t - 3t^2)\,x + (2t^3 - t^2) = 0$

Put $x = t$, we get $t^3 - t^2 + (2t - 3t^2)\,t + (2t^3 - t^2) = 0$

$\Rightarrow t^3 - t^2 + 2t^2 - 3t^3 + 2t^3 - t^2 = 0$

$\Rightarrow 3t^3 - 3t^3 + 2t^2 - 2t^2 = 0$

Hence, $x = t$ is the factor of above equation

$x^2(x - t) + x(t - 1)(x - t) - (2t^2 - t)(x - t) = 0$

$\Rightarrow [x^2 + x(t - 1) - (2t^2 - t)]\,(x - t) = 0$

$$\Rightarrow x=\frac{(1-t)\pm\sqrt{(1-t)^2+4(2t^2-t)}}{2\times 1}$$

$$\Rightarrow x=\frac{1-t\pm\sqrt{1+t^2-2t+8t^2-4t}}{2}$$

$$\Rightarrow x=\frac{1-t\pm\sqrt{9t^2-6t+1}}{2}=\frac{(1-t)\pm(1-3t)}{2}$$

$$\Rightarrow x=\frac{1-t+1-3t}{2}\text{ and }x=\frac{1-t-1+3t}{2}$$

$$\Rightarrow x=\frac{2-4t}{2}\text{ and }x=\frac{2t}{2}$$

$\Rightarrow x = 1 - 2t$ and $x = t \Rightarrow x = 1 - 2t$

2. (c) : The given curves are $x^2y = 1 - y$ ...(i)

and $xy = 1 - y$ ...(ii)

Solving (i) and (ii), we get

$x(1 - y) = (1 - y) \Rightarrow (1 - y)(x - 1) = 0 \Rightarrow x = 1, y = 1$

Now, it must satisfy equation of the curve $xy = 1 - y$

But, $1 \cdot 1 \neq 1 - 1$

If we take $x = 1$, $y = 0$, then $1\cdot 0 \neq 1 - 0$

If we take $x = 0$ and $y = 1$, then $0\cdot 1 = 1 - 1 = 0$

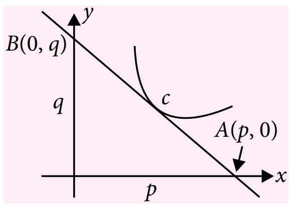
Hence, the required point is (0, 1)

3. (b) : We have $x^3 + y^3 = 2$

Differentiating w.r.t. x, we get

$3x^2 + 3y^2 \cdot dy/dx = 0$

$$\Rightarrow \frac{dy}{dx}=-\frac{x^2}{y^2}=-\left(\frac{x}{y}\right)^2 \quad ...(i)$$



Also, slope of tangent $= \dfrac{q-0}{0-p} = -\dfrac{q}{p}$ ...(ii)

From (i) and (ii), $\dfrac{x^2}{y^2} = \dfrac{q}{p} \Rightarrow x = y \times \sqrt{q/p}$ ...(iii)

Also, equation of tangent is

$\dfrac{x}{p} + \dfrac{y}{q} = 1 \Rightarrow qx + py = pq$

$\Rightarrow \quad q \times y\sqrt{q/p} + py = pq$ [from (iii)]

$\Rightarrow \quad y\left(\dfrac{q^{3/2}}{\sqrt{p}} + p\right) = pq$

$\Rightarrow \quad y(q^{3/2} + p^{3/2}) = p^{3/2}q \Rightarrow y = p^{3/2} \cdot q / (p^{3/2} + q^{3/2})$

$\Rightarrow \quad x = \dfrac{p^{3/2}.q}{(p^{3/2} + q^{3/2})} \times \sqrt{q/p} = pq^{3/2}/(p^{3/2} + q^{3/2})$

Put these values of x, y in equation of curve $x^3 + y^3 = 2$

$\Rightarrow \quad \dfrac{p^3.q^{9/2}}{(p^{3/2}+q^{3/2})^3} + \dfrac{q^3.p^{9/2}}{(p^{3/2}+q^{3/2})^3} = 2$

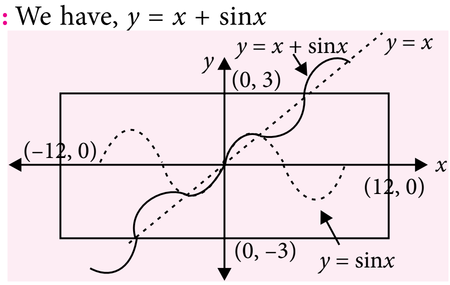
$\Rightarrow \quad \dfrac{p^3q^3[p^{3/2}+q^{3/2}]}{(p^{3/2}+q^{3/2})^3} = 2 \Rightarrow \dfrac{p^3q^3}{(p^{3/2}+q^{3/2})^2} = 2$

$\Rightarrow \quad \dfrac{(p^{3/2}+q^{3/2})^2}{(p^{3/2}q^{3/2})^2} = \dfrac{1}{2} \Rightarrow \dfrac{p^{3/2}+q^{3/2}}{p^{3/2}q^{3/2}} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \dfrac{1}{q^{3/2}} + \dfrac{1}{p^{3/2}} = \dfrac{1}{\sqrt{2}} \Rightarrow \dfrac{1}{p^{3/2}} + \dfrac{1}{q^{3/2}} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad p^{-3/2} + q^{-3/2} = \dfrac{1}{\sqrt{2}}$

4. (a) : We have, $y = x + \sin x$ ...(i)



Points on the curve at which tangents are parallel to x- axis *i.e.*, slope of tangents at that points must be zero.

Differentiating (i) w.r.t. x, we get

$\dfrac{dy}{dx} = 1 + \cos x \Rightarrow 1 + \cos x = 0$ $\left[\because \dfrac{dy}{dx} = 0\right]$

$\Rightarrow \quad \cos x = -1 = \cos\pi \quad \Rightarrow x = 2n\pi + \pi = (2n+1)\pi;\ n \in I$

If $n = 0$, $x = \pi$; If $n = -1$, $x = -\pi$

When $x = \pi \Rightarrow y = \pi + \sin\pi = \pi$ (outside rectangle)

When $x = -\pi \Rightarrow y = -\pi + \sin(-\pi) = -\pi$ (outside rectangle)

Hence, there is no points on the curve in the rectangle at which tangents are parallel to x- axis.

5. (c) : We have $y = ax^4 + bx^3 + cx + d$

$\Rightarrow \quad y' = 4ax^3 + 3bx^2 + c$

$\left(\dfrac{dy}{dx}\right)_{(0,1)} = 0 \Rightarrow c = 0$

Also, $\left(\dfrac{dy}{dx}\right)_{(-1,0)} = 0 \Rightarrow 3b - 4a = 0$

Given, $y' < 0$

$\Rightarrow \quad 4ax^3 + 3bx^2 < 0$

$\Rightarrow x^2(4ax + 3b) < 0 \Rightarrow x^2(x+1) < 0$ $[\because 3b = 4a]$

$\Rightarrow \quad x + 1 < 0 \quad \Rightarrow \quad x < -1$

6. (b) : At $x = 0$, $f(x) = -2 \Rightarrow d = -2$

At $x = -1$, $f(x) = 0$

$\Rightarrow \quad -a + b - c - 2 = 0$

$\Rightarrow \quad -a + b - c = 2$ ...(i)

At $x = 2$, $f(x) = 0$

$\Rightarrow \quad 8a + 4b + 2c = 2$

$\Rightarrow \quad 4a + 2b + c = 1$ ...(ii)

Also $3ax^2 + 2bx + c = 0$

At $x = 2$, $12a + 4b + c = 0$ ...(iii)

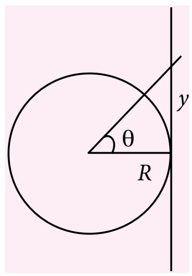
From (i) and (ii), $3a + 3b = 3$ ...(iv)

From (ii) and (iii), $8a + 2b = -1$ ...(v)

From (iv) and (v), $a = -1/2$, $b = 3/2 \Rightarrow c = 0$

Now, $a + b + c + d = -\dfrac{1}{2} + \dfrac{3}{2} + 0 + (-2) = -1$

7. (d) : Here, $R\tan\theta = y$



$\Rightarrow \quad R\sec^2\theta\left(\dfrac{d\theta}{dt}\right) = \dfrac{dy}{dt}$

Let $\omega = \dfrac{d\theta}{dt} \quad \therefore \quad (R\omega)\sec^2\theta = \dfrac{dy}{dt}$

Since horse is running at a speed of 20 km/hr.

$\therefore \quad \dfrac{dy}{dt} = 20 \times \sec^2\dfrac{\pi}{4}$ $\left[\because \theta = \dfrac{2\pi}{8}\right]$

$\Rightarrow \quad \dfrac{dy}{dt} = 20 \times 2 = 40$ km/hr.

8. (c) : $ax^{-2} + by^{-2} = 1$

Equation of tangent at (x_1, y_1) is

$ax \cdot x_1^{-3} + by \cdot y_1^{-3} = 1$

$\therefore \quad x\text{-intercept} = \dfrac{x_1^3}{a}$

9. (a) : $y' = \dfrac{a(2\cos t - 2\cos 2t)}{a(-2\sin t + 2\sin 2t)}$

$\Rightarrow$ $\cos t - \cos 2t = 0$

[$\because$ Tangents are parallel to x-axis]

$\Rightarrow$ $\cos 2t - \cos t = 0$

$\Rightarrow$ $2\cos^2 t - \cos t - 1 = 0$

$\Rightarrow$ $2\cos^2 t - 2\cos t + \cos t - 1 = 0$

$\Rightarrow$ $(2\cos t + 1)(\cos t - 1) = 0$

$\Rightarrow$ $\cos t = 1,\ \cos t = -\dfrac{1}{2}$

$\Rightarrow$ $t = 0,\quad t = \dfrac{2\pi}{3}$

Exclude the point where denominator of y' becomes zero.

$\therefore$ $t = 2\pi/3$

10. (d) : $f(x)$ is continuous and differentiable $\forall\ x \in R$.

Rolle's theorem is not applicable, as

$f(-1) \neq f(2)$

Now, by LMVT, $3c^2 = \dfrac{5}{3} \Rightarrow c^2 = \dfrac{5}{9} \Rightarrow c = \pm\dfrac{\sqrt{5}}{3}$

11. (d) : $f(0) = 1, f(1) = 1$

$f'(x) = 0$ must have at least one root in $(0, 1)$

$f'(x) = x^{m-1} \cdot m(x-1)^n + nx^m(x-1)^{n-1}$

$= x^{m-1}(x-1)^{n-1}[(m+n)x - m]$

$\Rightarrow$ $x = 0,\ 1, \left(\dfrac{m}{m+n}\right)$

12. (c) : Here, $f(x) = \left(\dfrac{\sqrt{3a-5}}{1-a} - 1\right)x^5 - 3x + \log 3$ ...(i)

Differentiating (i) w.r.t. x, we get

$$f'(x) = \left(\dfrac{\sqrt{3a-5}}{1-a} - 1\right)5x^4 - 3$$

Now, $\left(\dfrac{\sqrt{3a-5}}{1-a} - 1\right)5x^4 - 3 < 0$

[$\because f(x)$ is decreasing function]

$\Rightarrow$ $\left(\dfrac{\sqrt{3a-5}}{1-a} - 1\right)x^4 < 3/5\ \forall x \in R$

It must be $3a - 5 \geq 0;\ a \neq 1$ $\Rightarrow$ $a \geq 5/3$; $a \neq 1$

Hence, $a \in \left[\dfrac{5}{3}, \infty\right)$

13. (b) : Here, $f(x) = \dfrac{\lambda \sin x + 3\cos x}{2\sin x + 6\cos x}$

Differentiating w.r.t. x, we get

$$f'(x) = \dfrac{[(\lambda\cos x - 3\sin x)(2\sin x + 6\cos x) - (\lambda \sin x + 3\cos x)(2\cos x - 6\sin x)]}{(2\sin x + 6\cos x)^2}$$

Since, $f(x)$ is monotonically increasing function

i.e., $f'(x) > 0$

$\Rightarrow$ $[2\lambda \sin x \cos x - 6\sin^2 x + 6\lambda\cos^2 x - 18\sin x \cos x]$

$-[2\lambda \sin x \cos x + 6\cos^2 x - 6\lambda \sin^2 x - 18 \sin x \cos x] > 0$

$\Rightarrow$ $-6 + 6\lambda\ (1) > 0$ $\Rightarrow$ $\lambda > 1$

14. (b) : Let $\phi(x) = f^2(x)$

Differentiating w.r.t. x, we get

$\phi'(x) = 2f(x).\ f'(x) < 0;\ x \in R$

It means, $\phi(x)$ is decreasing function for all $x \in R$.

15. (b) : Here, $y = \dfrac{x\sqrt{x}}{\sqrt{1-x}}$ $\forall\ x \in (0, 1)$

$$\text{Now, } y' = \dfrac{\dfrac{-1-3t^2}{t^2(1+t^2)^2}}{\dfrac{-2t}{(1+t^2)^2}} = \dfrac{3t^2+1}{2t^3}$$

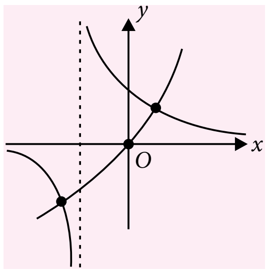
Since, $t > 0$

$\therefore$ $y' > 0$

So, $f(x)$ is increasing in $(0, 1)$.

16. (c) : $f(x) = \dfrac{1}{(x+1)^3} - 3x + \sin x = 0$

$\Rightarrow$ $\dfrac{1}{(x+1)^3} = 3x - \sin x$



Thus, $f(x)$ has 2 roots.

17. (a) : When $t \in [-1, 0]$

$f'(t) = \dfrac{f(0) - f(-1)}{1}$ ...(i)

When $t \in\ [0, 2]$

$f'(t) = \dfrac{f(2) - f(0)}{2}$ ...(ii)

(i) $\Rightarrow$ $0 \leq f(0) - f(-1) \leq 1$

(ii) $\Rightarrow$ $-2 \leq f(2) - f(0) \leq 0$

On adding, we get $-2 \leq f(2) - f(-1) \leq 1$

18. (b) : We have, $g(x) = f(\sin x) + f(\cos x)$

$\Rightarrow \quad g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

Now, $f'(\sin x) < 0 \Rightarrow f'(\cos x) < 0$

and $f''(\sin x) > 0 \Rightarrow f''(\cos x) > 0$

Also, $g''(x) = f''(\sin x)\cos^2 x - f'(\sin x)\sin x$
$\quad -f'(\cos x)\cos x + f''(\cos x)\sin^2 x$

$\Rightarrow \quad g''(x) > 0$

For critical point of $g(x)$, put $g'(x) = 0$

$$\Rightarrow \quad x = \frac{\pi}{4}$$

So, $x = \frac{\pi}{4}$ is a point of minima.

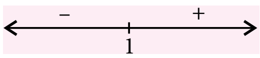
So, $g(x)$ is decreasing in $\left(0, \frac{\pi}{4}\right)$ and increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

19. (a) : $f(x) > 0$ for $x = 0^+$ and $x = 0^-$

$\therefore \quad f(x)$ is min $= 0$ at $x = 0$

20. (a) : $f'(x) = x^{p-1} - x^{-q-1} = x^{-q-1}(x^{p+q} - 1)$

$$f_{\min} = f(1) = \frac{1}{p} + \frac{1}{q} = 1$$



21. (b) : $f(x) = 4x - (a + b + c + d);\ x \geq d$

$= 2x - (a + b + c - d);\ c \leq x < d$

$= -(a + b - c - d);\ b \leq x < c$

$= -2x - (a - b - c - d)\ ;\ a \leq x < b$

$= (a + b + c + d) - 4x\ ;\ x < a$

$f_{\min} = -a - b + c + d$

22. (c) : $f'(x) = 3x^2 - 6px + 3(p^2 - 1) = 3[(p - x)^2 - 1]$

(1) $D > 0 \Rightarrow 36p^2 - 36(p^2 - 1) > 0 \Rightarrow 1 > 0 \Rightarrow p \in R$

(2) $-2 < p < 4$

(3) $f(-2) > 0 \Rightarrow -8 - 12p - 6p^2 + 6 + 1 > 0$

$\Rightarrow \quad -6p^2 - 12p - 1 > 0 \Rightarrow 6p^2 + 12 + 1 < 0 \Rightarrow p \in R$

(4) $f(4) > 0 \Rightarrow 64 - 48p + 12p^2 - 12 + 1 > 0$

$\Rightarrow \quad 12p^2 - 48p + 53 > 0 \Rightarrow p \in R$

$\therefore \quad p \in (-2, 4)$ or $f'(x) = 0$

$\Rightarrow \quad x = p - 1, p + 1$

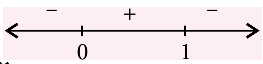
$\quad p - 1 \in (-2, 4) \Rightarrow p \in (-1, 5)$

$\quad p + 1 \in (-2, 4) \Rightarrow p \in (-3, 3)$

$\therefore \quad p \in (-1, 3)$

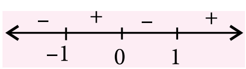
23. (a) : Let $f(x) = 3x^2 - 2x^3$

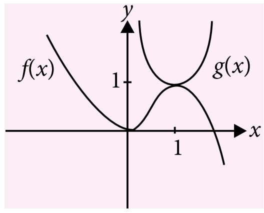
$f'(x) = 6x - 6x^2 = 6x(1 - x)$



Let $g(x) = \log_2(x^2 + 1) - \log_2 x$

$$g'(x) = \frac{1}{\log 2} \times \frac{2x}{(x^2+1)} - \frac{1}{x\log 2} = \frac{1}{\log 2} \times \frac{x^2 - 1}{x(x^2+1)}$$





Thus, only one solution exists.

24. (a) : Equation of line through (3, 4) is

$y - 4 = m(x - 3)$

Now, x-intercept $= 3 - \frac{4}{m}$, y-intercept $= 4 - 3m$

$$\text{Now, } A = \frac{1}{2}(4 - 3m)\left(3 - \frac{4}{m}\right) = \frac{1}{2}\left(24 - 9m - \frac{16}{m}\right)$$

For $A_{\min}$, $9m = \frac{16}{m} \Rightarrow m = -\frac{4}{3}$

Thus, required line is $4x + 3y - 24 = 0$

25. (d) : Let $f(x) = a\tan^2\theta + b\cot^2\theta$

Now, A.M. $\geq$ G.M. $\Rightarrow f(x) \geq 2\sqrt{ab}$

Let $g(x) = a\sin^2\theta + b\cos^2\theta = (a - b)\sin^2\theta + b$

$\quad g(x)_{\max} = a$

Now, $a = 2\sqrt{ab} \Rightarrow a^2 = 4ab$

$\Rightarrow \quad a(a - 4b) = 0 \Rightarrow a = 4b$

26. (d) : $g'(x) = \dfrac{h'(x)}{h(x)}$

$$g''(x) = \frac{h(x)h''(x) - [h'(x)]^2}{[h(x)]^2} < 0$$

$\therefore \quad g(x)$ is concave down on J.

27. (c)

28. (a) : $f'(a) = 4a^3 - 3a^2 - 2a + a = 0$

$\Rightarrow \quad a = 0, 1, -1/4$

Now, $f''(a) = 12a^2 - 6a - 1$

$$f''(0) = -1,\ f''(1) = 5,\ f''(-1/4) = \frac{5}{4}$$

So, $a = 0$ is not possible for minima.

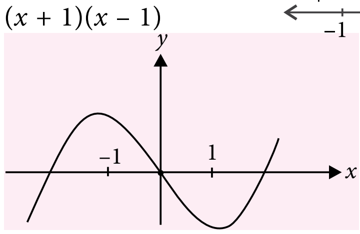
So, $f(a) = a \Rightarrow a = 1$

29. (d) : $f'(x) = \sin a\tan^2 x\sec^2 x + (\sin a - 1)\sec^2 x$

$\quad = \sin a\sec^4 x - \sec^2 x = \sec^2 x(\sin a\sec^2 x - 1)$

30. (c) : $f'(x) = 3ax^2 - 3a$

$\quad = 3a(x + 1)(x - 1)$

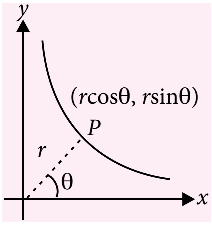


31. (b) : $5r^2 - 6r^2 \sin\theta \cos\theta = 4$

$$\Rightarrow \quad r^2 = \frac{4}{5 - 3\sin 2\theta}$$

$$\Rightarrow \quad r_{\min}^2 = \frac{4}{8} = \frac{1}{2} \quad \Rightarrow \quad r_{\min} = \frac{1}{\sqrt{2}}$$

When $2\theta = \frac{-\pi}{2}, \frac{3\pi}{2}; \quad \theta = \frac{-\pi}{4}, \frac{3\pi}{4}$

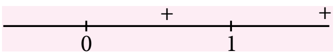


32. (a) : $f(x) = \int_{x^2}^{x^3} \frac{dt}{\ln t}$

$$\Rightarrow \quad f'(x) = \frac{1}{\ln x^3} \cdot 3x^2 - \frac{1}{\ln x^2} \cdot 2x$$

(Using Leibnitz formula)

$$= \frac{1}{\ln x}(x^2 - x)$$



Since $f'(x) > 0$ for $x > 0$, $x \neq 1$. Hence $f(x)$ is increasing function.

33. (b) : Let $g(x) = f(f(x))$

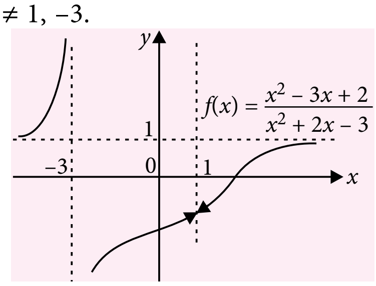
$g'(x) = f'(f(x))\, f'(x) < 0$

$g(1) = f(f(1)) = f(1) = 1; \quad g(x) \leq 1$

Also, $\frac{1}{(x-1)^2 + 1} \leq 1$

34. (c) : $f(x) = \frac{x^2 - 3x + 2}{x^2 + 2x - 3} = \frac{(x-1)(x-2)}{(x-1)(x+3)} = \frac{x-2}{x+3}$,

$x \neq 1, -3$.



$$\frac{df}{dx} = \frac{(x+3) - (x-2)}{(x+3)^2} = \frac{5}{(x+3)^2} > 0 \ \forall\, x \neq 1, -3$$

Clearly, $f(x)$ is increasing in its domain.

35. (a) : $\frac{N}{x_1 + x_2 + x_3 + x_4} = \frac{1000x_1 + 100x_2 + 10x_3 + x_4}{x_1 + x_2 + x_3 + x_4}$

$$= 1000 - \frac{(900x_2 + 990x_3 + 999x_4)}{(x_1 + x_2 + x_3 + x_4)}$$

$\Rightarrow$ Maximum value of $\frac{N}{x_1 + x_2 + x_3 + x_4} = 1000$

36. (b) : Let $f(x) = x^5 - 10a^3x^2 + b^4x + c^5$

$\Rightarrow \ f'(x) = 5x^4 - 20a^3x + b^4 \Rightarrow f''(x) = 20x^3 - 20a^3$

If $x = \alpha$ be a root that is repeated three times, then

$f''(\alpha) = 0, f'(\alpha) = 0, f(\alpha) = 0$

$\Rightarrow \ \alpha = a, 5a^4 - 20a^4 + b^4 = 0, a^5 - 10a^5 + ab^4 + c^5 = 0$

$\Rightarrow \ \alpha = a, b^4 = 15a^4, c^5 + ab^4 - 9a^5 = 0$

$\Rightarrow \ c^5 + 15a^5 - 9a^5 = 0 \Rightarrow 6a^5 + c^5 = 0.$

37. (c) : Since roots are positive $\Rightarrow$ A.M. $\geq$ G. M. can be applied.

Let roots are α_1, α_2 and α_3

$$\Rightarrow \quad \frac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1}{3} \geq ((\alpha_1\alpha_2\alpha_3)^2)^{1/3}$$

$$\Rightarrow \quad \frac{b}{3} \geq (36)^{1/3} \quad \Rightarrow b \geq 3(36)^{1/3}$$

Hence, $b_{\min} = 3(36)^{1/3}$

38. (a) : At point of maxima $f'(x) = 0$ and $f''(x) < 0$

$\Rightarrow \ f''(x) = x^2 - f^2(x) \leq 0$

$\Rightarrow$ Point $P(x, f(x))$ lies inside $x^2 - y^2 = a^2$.

Hence, two normals can be drawn.

39. (b) : $g'(\theta) = (f(\sin^2\theta) - f(\cos^2\theta))\sin 2\theta$

For $g(\theta)$ to be increasing $g'(\theta) > 0$

$\Rightarrow \ f(\sin^2\theta) > f(\cos^2\theta) \quad \left[\because \sin 2\theta > 0 \ \forall \ \theta \in \left(0, \frac{\pi}{2}\right)\right]$

$\Rightarrow \ \sin^2\theta > \cos^2\theta$

$$\Rightarrow \quad \tan^2\theta > 1 \quad \Rightarrow \quad \theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

40. (a) : The given expression resembles with $(x_1 - x_2)^2 + (y_1 - y_2)^2$; where $y_1 = \frac{x_1^2}{20}$ and $y_2 = \sqrt{(17 - x_2)(x_2 - 13)}$

Let us consider two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on the curves $x^2 = 20y$ and $(x - 15)^2 + y^2 = 4$. If D be the distance between P_1 and P_2, then given expression represents D^2.

Now D has to be minimized as per the problem.

Since shortest distance between two curves always occur along the common normal, it implies that $P_1(x_1, y_1)$ lies on $x^2 = 20y$ such that the normal at this point passes through $(15, 0)$.

Normal to parabola is $y - \frac{x_1^2}{20} = \frac{-10}{x_1}(x - x_1)$

It should pass through $(15, 0)$.

$\therefore \ x_1^3 + 200x_1 - 3000 = 0 \ \Rightarrow \ x_1 = 10 \ \& \ y_1 = 5$

$$\Rightarrow \quad D = \sqrt{(10-15)^2 + 5^2} - 2 = 5\sqrt{2} - 2$$

Thus, minimum value of given expression is $(5\sqrt{2} - 2)^2$.

MATHS MUSING

SOLUTION SET-189

1. (a) : $\sqrt{3+4i} = 2+i,\ \sqrt{-3-4i} = 1-2i$
$z = 2 + i + 1 - 2i = 3 - i$
$\therefore$ Re$(z) = 3$.

2. (d) : $\because$ a, b, c are in A.P. and $a+b+c=\dfrac{3}{2}$

$\Rightarrow 3b=\dfrac{3}{2} \Rightarrow b=\dfrac{1}{2}$

$\therefore a=\dfrac{1}{2}-d,\ c=\dfrac{1}{2}+d,$

Now, a^2, b^2, c^2 are in G.P.

$\Rightarrow a^2c^2=b^4=\dfrac{1}{16} \Rightarrow ac=\pm\dfrac{1}{4} \Rightarrow \dfrac{1}{4}-d^2=\pm\dfrac{1}{4}$

If $ac=\dfrac{1}{4}$, $d=0$; $a=b=c$

and if $ac=-\dfrac{1}{4} \Rightarrow d^2=\dfrac{1}{2} \Rightarrow d=\dfrac{1}{\sqrt{2}}$

$\therefore a=\dfrac{1}{2}-d=\dfrac{1}{2}-\dfrac{1}{\sqrt{2}}$

3. (b) : Required Probability

$=\binom{7}{3}4!\left(\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}\right)\div 7!=\dfrac{1}{3!}\left(\dfrac{1}{2}-\dfrac{1}{6}+\dfrac{1}{24}\right)=\dfrac{1}{16}$

4. (b) : Replacing $f(x)$ by x and x by $g(x)$.

$x=\displaystyle\int_0^{g(x)}\dfrac{dt}{\sqrt{1+t^3}}$. Differentiating, $1=\dfrac{g'}{\sqrt{1+g^3}}$

or $(g')^2 = 1 + g^3$. Differentiating, $g''=\dfrac{3}{2}g^2$.

5. (a) : We have,

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}=\begin{bmatrix}1&3\\0&1\end{bmatrix}=\begin{bmatrix}1&1+2\\0&1\end{bmatrix}$$

Again $\begin{bmatrix}1&3\\0&1\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}=\begin{bmatrix}1&6\\0&1\end{bmatrix}=\begin{bmatrix}1&1+2+3\\0&1\end{bmatrix}$

By induction, L.H.S. $=\begin{bmatrix}1&\sum\limits_{k=1}^{n}k\\0&1\end{bmatrix}=\begin{bmatrix}1&378\\0&1\end{bmatrix}$

Therefore, $\dfrac{n(n+1)}{2}=378$ or $n(n+1) = 27 \times 28$

Hence, $n = 27$.

6. (b, c) : $3=\dfrac{2\sin(A+C)\sin C}{\cos A}$

$= 2(\tan A \cos C + \sin C) \sin C = 2(4 \cos^2 C + 1) \sin^2 C$
$\therefore$ $3 = 2s(5 - 4s)$, where $s = \sin^2 C$
Solving $8s^2 - 10s + 3 = 0$, we get

$\sin^2 C=s=\dfrac{3}{4},\dfrac{1}{2}\ \therefore\ C=\dfrac{\pi}{3},\dfrac{\pi}{4}.$

7. (d) : $S \equiv (a, 0)$, $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$, $C \equiv (at_3^2, 2at_3)$

$SA \perp BC \Rightarrow \left(\dfrac{2t_1}{t_1^2-1}\right)\left(\dfrac{2}{t_2+t_3}\right)=-1$

$\Rightarrow 4t_1 + t_1^2(t_2 + t_3) = t_2 + t_3$ ...(i)

Likewise, $4t_2 + t_2^2(t_3 + t_1) = t_3 + t_1$ ...(ii)

From (i) and (ii), we get $4 + \Sigma t_1t_2 = -1$
$\therefore$ $\Sigma t_1t_2 = -5$
Now (i) gives $t_1[4 - 5 - t_2t_3] = t_2 + t_3$
$\Rightarrow t_1 + t_2 + t_3 + t_1t_2t_3 = 0$.

8. (d) : $(t_1 - 1)(t_2 - 1)(t_3 - 1)$
$= t_1t_2t_3 - \Sigma t_1t_2 + \Sigma t_1 - 1 = 0 + 5 - 1 = 4$.

9. (3) : If $(x - x_1)^2 + (y - y_1)^2 = r^2$ touches $(x - 2)^2 + y^2 = 1$ and $x^2 + (y - 3)^2 = 4$ externally, then

$\sqrt{(x_1-2)^2+y_1^2}=r+1,\ \sqrt{x_1^2+(y_1-3)^2}=r+2$

$\therefore$ The locus of $P(x_1, y_1)$ is given by

$\sqrt{(x-2)^2+y^2}+1=\sqrt{x^2+(y-3)^2}$

Squaring, $\sqrt{(x-2)^2+y^2}=2+2x-3y$

Squaring again, $3x^2 + 8y^2 - 12xy + 12x - 12y = 0$

$\dfrac{3}{4}x^2+2y^2-3xy=3(y-x) \Rightarrow \lambda = 3.$

10. (d) : (P) $\to$ 1, (Q) $\to$ 4, (R) $\to$ 4, (S) $\to$ 3

(P) $\displaystyle\lim_{x\to0}\dfrac{\tan x^2}{x\sin x}=\lim_{x\to0}\dfrac{\tan x^2}{x^2}\cdot\dfrac{x}{\sin x}=1\times1=1$

(Q) $I=2\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\sec x-\cos x}\,dx=2\int_0^{\frac{\pi}{2}}\dfrac{\sin x\,dx}{\sqrt{\cos x}}$

$=-4\sqrt{\cos x}\Big|_0^{\frac{\pi}{2}}=4$

(R) $I=\displaystyle\int_1^2\dfrac{2}{x}e^{\sin x^2}\cdot dx,\ x^2=t$

$=\displaystyle\int_1^4\dfrac{e^{\sin t}}{t}dt=F(4)-F(1)\Rightarrow k=4.$

(S) $y^2=2c\left(x+\sqrt{c}\right) \Rightarrow 2yy_1 = 2c \Rightarrow c = yy_1$

Eliminating c, $y^2=2yy_1\left(x+\sqrt{yy_1}\right)$

$y-2xy_1=2y_1\sqrt{yy_1}$

$(y - 2xy_1)^2 = 4yy_1^3 \Rightarrow$ degree = 3.

TARGET JEE

Complex Numbers and Quadratic Equations

QUADRATIC EQUATIONS

An equation of the form $ax^2 + bx + c = 0 \ \forall \ a, b, c \in R$, $a \neq 0$ is called a quadratic equation. If α, β are its roots and we recall $D = b^2 - 4ac$, known as discriminant of the equation, then the values of α and β are given by

$$\alpha = \frac{-b+\sqrt{D}}{2a},\ \beta = \frac{-b-\sqrt{D}}{2a} \text{ or vice-versa.}$$

Nature of Roots

The nature of the roots of equation $ax^2 + bx + c = 0$ ...(*) depends on the following facts:

(i) If $D = b^2 - 4ac < 0$, then roots of the equation (*) be non-real complex roots. The complex roots always appear in pair and they are conjugate of each other. The complex roots α, β are given by $\alpha = \dfrac{-b+i\sqrt{|D|}}{2a}$ and $\beta = \dfrac{-b-i\sqrt{|D|}}{2a}$

(ii) If $D = 0$, then roots of the equation are real and equal *i.e.*, $\alpha = \beta = \dfrac{-b}{2a}$ and we can express $ax^2 + bx + c = 0$ as $a(x - \alpha)^2 = 0$.

(iii) If $D > 0$, then roots of the equation be real and distinct *i.e.*, $\alpha = \dfrac{-b+\sqrt{|D|}}{2a}$ and $\beta = \dfrac{-b-\sqrt{|D|}}{2a}$.

(iv) If D is a perfect square, then roots of equation (*) are rational. *e.g.*, the equation $30x^2 + 16x + 2 = 0$ has roots $-\dfrac{1}{3}, -\dfrac{1}{5}$ which are rational.

(v) If D is non perfect square, then roots of equation (*) are irrational and conjugate of each other *i.e.*, if one root is $p + \sqrt{q}$, then other root be $p - \sqrt{q}$ where p is a rational and q is a non-zero surd.

(vi) If the equation $ax^2 + bx + c = 0$, where $a, b, c \in R$, has more than two roots (real or complex), then it becomes an identity *i.e.*, $a = b = c = 0$.

Nature of Roots of $f(x) \cdot g(x) = 0$

If D_1 and D_2 are the discriminants of quadratic equations $f(x) = 0$ and $g(x) = 0$ respectively, then we have the following (cases) possibilities arise about the roots of $f(x) \cdot g(x) = 0$

(i) If $D_1 + D_2 \geq 0$, then there will be at least two real roots of the equation $f(x) \cdot g(x) = 0$.

(ii) If $D_1 + D_2 < 0$, then there will be at least two imaginary roots.

(iii) If $D_1D_2 < 0$, then there will be two real roots.

(iv) If $D_1D_2 > 0$, then the equation has either all roots real or no real roots (*i.e.*, all roots be imaginary).

Symmetric Functions

If $f(\alpha, \beta) = f(\beta, \alpha)$, then f is called symmetric function. If f is symmetric, then λf is also symmetric ($\lambda \in R$). The sum, difference, product and quotient of two symmetric functions are also symmetric.

Let α, β are roots of the quadratic equation $ax^2 + bx + c = 0$ ($\forall\ a \neq 0$), then $S_1 = \alpha^1 + \beta^1$, $S_2 = \alpha^2 + \beta^2$, ... , $S_n = \alpha^n + \beta^n$ are known as symmetric functions.

- To determine the sum of the symmetric roots of $ax^2 + bx + c = 0$, we use the recurrence relation formula

$$S_{n+1} = -\frac{b}{a}S_n - \frac{c}{a}S_{n-1} \quad \text{...(*)}$$

- To determine the sum of the symmetric roots of cubic equation $ax^3 + bx^2 + cx + d = 0$, we use the recurrence formula

$$S_{n+1} = -\frac{b}{a}S_n - \frac{c}{a}S_{n-1} - \frac{d}{a}S_{n-2}$$

or $$S_{n+2} = -\frac{b}{a}S_{n+1} - \frac{c}{a}S_n - \frac{d}{a}S_{n-1}$$

(Replacing n by $n + 1$ in above formula)

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For a quadratic equation $ax^2 + bx + c = 0$, if we need $\alpha^3 + \beta^3$, then from [*] $\alpha^3 + \beta^3 = S_3$

$$= -\frac{b}{a}S_2 - \frac{c}{a}S_1 \quad \text{(Putting } n = 2\text{)}$$

$$= -\frac{b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right], \text{ where } S_1 = \alpha + \beta = -b/a \text{ and}$$

$$S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a}$$

- Let $ax^2 + bx + c = 0$ have roots α, β & we need new equation whose roots are $2\alpha, 2\beta$, then replace x by $\frac{x}{2}$ and if new equation have thrice the roots of $ax^2 + bx + c = 0$, then replace x by $\frac{x}{3}$.

EQUATIONS REDUCES TO LINEAR, QUADRATIC AND BIQUADRATIC EQUATIONS

- **Type 1:** An equation of the form $(x - a)(x - b)(x - c)(x - d) = A$, $a < b < c < d$ and $b - a = d - c$, then to reduce in simple form by taking
$$y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$$
or $$y = x - \frac{(a+b+c+d)}{4}$$
- **Type 2:** An equation of the form $(x - a)(x - b)(x - c)(x - d) = Ax^2$, if $ab = cd$, can be reduced to a collection of two quadratic equation by keeping $y = x + \frac{ab}{x}$.
- **Type 3:** An equation of the form $(x \pm a)^4 + (x \pm b)^4 = c$ can be reduces to simple form of biquadratic by making substitution, $t = \frac{(x\pm a)+(x\pm b)}{2}$ or $t = x \pm \frac{a+b}{2}$.

Note: (i) If α is the repeated root repeating r times of a polynomial equation $f(x) = 0$ of degree n, then we can write $f(x) = (x - \alpha)^r g(x)$ where $g(x)$ is a polynomial of degree $(n - r)$ and $g(x) \neq 0$ then we have $f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{r-1}(\alpha) = 0$ & $f^r(\alpha) \neq 0$.
(ii) If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are real numbers and $g(x)$ be a polynomial function of degree n such that $f(\alpha_i) = 0$, then α_i is called the zero of the polynomial and $(x - \alpha_i)$ is called the factor of the polynomial function $f(x)$.

TRANSFORMATION OF EQUATIONS

Let $ax^2 + bx + c = 0$ be given quadratic equation (the equation may be cubic, biquadratic or higher order) and we need new equation whose roots are negative and reciprocal of the roots of the original equation $ax^2 + bx + c = 0$, then to obtain new equation replace x by $\left(-\frac{1}{x}\right)$. The new equation be $a\left(-\frac{1}{x}\right)^2 + b\left(-\frac{1}{x}\right) + c = 0$ *i.e.*, $cx^2 - bx + a = 0$. The roots of $f(x) = 0$ and $f\left(-\frac{1}{x}\right) = 0$ are negative and reciprocal of each other.

- The roots of $f(x) = 0$ and $f(-x) = 0$ are equal in magnitude but opposite in sign *i.e.*, if $ax^2 + bx + c = 0$ be a equation whose roots are α, β, then the equation $a(-x)^2 + b(-x) + c = 0$ have roots $-\alpha, -\beta$.
- If the roots of a new equation are squares of the roots of original equation, then to obtain new equation replace x by $x^{1/2}$ in the original equation and if roots of new equations are cubes of original equation then replace x by $x^{1/3}$ in the original equation.
- If the roots of a new equation are one more than the roots of original equation, then replace x by $x - 1$ in the original equation.
- If the roots of a new equation are one less than the roots of original equation $ax^2 + bx + c = 0$, then replace x by $x + 1$ in $ax^2 + bx + c = 0$ & the new equation be $a(x + 1)^2 + b(x + 1) + c = 0$.

Important Points

(1) For a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ $a, b, c \in R$ ...(*)
 (a) If constant term is 0 *i.e.*, $c = 0$, then one of its root is 0.
 (b) If $b = c = 0$, then both roots of equations are zero.

(2) If ratio of roots of quadratic equation $ax^2 + bx + c = 0$ is $p : q$, then $pqb^2 = (p + q)^2 \cdot c$ which is used to find the unknown when ratio of the roots of equation given.

(3) If one root of the equation $ax^2 + bx + c = 0$ is k times the other root, then to determine unknown quantity use the result $kb^2 = ac(k + 1)^2$.

(4) For a quadratic equation $ax^2 + bx + c = 0$ if $a + b + c = 0$, then its roots are 1 and c/a *i.e.*, 1 and $\frac{\text{constant term}}{\text{coefficient of } x^2}$.
 (a) If $ab + bc + ca = 0$, then for quadratic equation $a(b - 2c)x^2 + b(c - 2a)x + c(a - 2b) = 0$, then roots of equation are 1 and $\frac{c(a-2b)}{a(b-2c)}$.
 (b) For a quadratic equation $f(x) = ax^2 + bx + c = 0$, $f(-1) = 0$ *i.e.*, $a - b + c = 0$, then roots of equation are -1 and $-\frac{c}{a}$ $(a \neq 0)$.

(5) If roots of the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, $a, b, c \in R$ are reciprocal to each other, then product of roots is always 1 which means constant term and coefficient of x^2 are identical *i.e.*, $c = a$.

(6) If one root of the equation $ax^2 + bx + c = 0$ $(a \neq 0, a, b, c \in R)$ is n^{th} power of the other root, then $(ac^n)^{\frac{n+1}{n}} + (a^n c)^{\frac{n+1}{n}} + b = 0$.

(7) If the roots of equations $ax^2 + bx + c = 0$ $(a \neq 0)$ and $a'x^2 + b'x + c' = 0$, $[a' \neq 0]$ are reciprocal to each other, then $(aa' - cc')^2 = (ba' - cb')(ab' - bc')$.

(8) If α and β are roots of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then equation

(a) whose roots are $\alpha \pm k$, $\beta \pm k$ is $a(x \pm k)^2 + b(x \pm k) + c = 0$.

(b) whose roots are $\frac{\alpha}{k}, \frac{\beta}{k}$ is $ak^2x^2 + bkx + c = 0$

(c) whose roots are αk, βk is $ax^2 + bkx + k^2c = 0$

(d) whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ is obtained by interchanging constant term & coefficient of x^2 & new equation is $cx^2 + bx + a = 0$.

(9) (a) If in a quadratic equation $ax^2 + bx + c = 0$, $a > 0$, $D = b^2 - 4ac < 0$, then the expression $f(x) = ax^2 + bx + c$ is always > 0.

(b) If in a quadratic equation $ax^2 + bx + c = 0$, $a < 0$, $D = b^2 - 4ac < 0$, then the expression $f(x) = ax^2 + bx + c$ is always < 0.

(c) If the sum of the coefficient of a polynomial equation $b_0 + b_1x + b_2x^2 + \ldots + b_nx^n = 0$ is zero, then $x = 1$ is always a root of the equation.

(10) (a) If in a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0, a, b\ c \in R)$, the sign of b is opposite of sign a and c then both roots of equations are > 0.

(b) If a, b, c are all of same sign, then roots of the equation $ax^2 + bx + c = 0$ are less than zero *i.e.*, negative.

(c) If in quadratic equation $ax^2 + bx + c = 0$, $b = 0$ and $c < 0$, then roots of equation are irrational and they are irrational conjugate of each other.

(d) If $b = 0$ and $c > 0$, then equation $ax^2 + bx + c = 0$ has complex *i.e.*, imaginary roots and they will be conjugate of each other whose magnitudes are equal.

(11) If $f(x) = ax^2 + bx + c = 0$ and $f(\alpha) = 0$, $f'(\alpha) = 0$, then α is repeated root of $f(x) = 0$ and we can write $f(x) = a(x - \alpha)^2$ and value of $\alpha = -b/2a$.

(12) If both roots of equations $ax^2 + bx + c = 0$ $(a \neq 0)$ and $a'x^2 + b'x + c' = 0$ $(a' \neq 0)$ are common, then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

- If in a cubic equation $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$ two roots are equal in magnitude but opposite in sign, then third root must be $= \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$ and the condition that roots are equal but opposite in sign is $ad = bc$.
- If a, b, c are consecutive integers then polynomial equation $f(x) = (x - a)^2 + (x - b)^2 + (x - c)^2 = 0$, have no real roots, both roots should be imaginary and conjugate of each other, here $f(x) > 0$.
- Any cubic equation of the form $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$, where a, b, c are consecutive integer, then the equation has one real and two imaginary roots as $f'(x) = 3(x - a)^2 + 3(x - b)^2 + 3(x - c)^2 > 0$ and have no real roots.
- If a cubic equation $ax^3 + bx^2 + cx + d = 0$ have all roots equal, it means roots are in A.P. and G.P. both. The condition for equal roots is $b^3d = ac^3$ for cubic equation.
- Let $ax^2 + bx + c = 0$ $\forall$ $(a \neq 0, a, b, c \in R)$. If roots of the equation are real and distinct then both roots are less than negative of the average of their sum *i.e.*, if α, β are real roots $(\alpha \neq \beta)$ of the equation $ax^2 + bx + c = 0$, then α, β both $< -b/2a$.
- If $a, b, c > 0$, then equation $a|x|^2 + b|x| + c = 0$ have no real roots in this case both roots will be imaginary and conjugate of each other *e.g.*, $x^2 + 5|x| + 4 = 0$ have real root because we can write the equation as $|x|^2 + 5|x| + 4 = 0 \Rightarrow (|x| + 1)(|x| + 4) = 0$ $\Rightarrow |x| = -1, |x| = -4$ which is impossible if $x \in R$.
- If $f(x) = ax^2 + bx + c = 0$ $(a \neq 0, a, b, c \in R)$ and $a + b + c < 0$ and equation have no real roots, then $c < 0$.
- If both roots of the equation $ax^2 + bx + c = 0$ $(a > 0)$ are greater than 1, then $a + b + c > 0$
- If two roots of the equation $ax^3 + bx^2 + cx + d = 0$ are equal in magnitude but opposite in sign, then, $ab = c$.
- Let $f(x)$ be a polynomial of degree 1, 2, ..., n and $f(-x) = f(x)$ means polynomial is even function then all roots of the polynomial are imaginary and we know that imaginary roots always occurs in pair and conjugate of each other.
- Quadratic equation whose roots are A.M. and H.M. between the roots of the equation $ax^2 + bx + c = 0$ is given by $x^2 - (\text{A.M.} + \text{H.M.})x + \text{A.M}\cdot\text{H.M.} = 0$. Let α, β are roots of $ax^2 + bx + c = 0$, then A.M. $= \frac{\alpha+\beta}{2} = -\frac{b}{2a}$, H.M. $= \frac{2\alpha\beta}{\alpha+\beta} = \frac{2c}{b}$.

$\therefore$ Required equation is $abx^2 + (b^2 + 4ac)x + 2bc = 0$.

COMPLEX NUMBERS

Definition : Any number which can be expressed in the form $a + ib$ is called a complex number. We denote the complex number by c or z.

$\therefore$ $z = a + ib$ $(a, b \in R)$ where 'a' is called the real part of z and 'b' is called the imaginary part of z. If $z = a + ib$, then we write $\text{Re}(z) = a$ and $\text{Im}(z) = b$. If $z = a + ib$ $(a, b \in R)$ we denote z by an ordered pair (a, b).

PURELY REAL AND PURELY IMAGINARY COMPLEX NUMBERS

A complex number is called purely real if its imaginary part is zero *i.e.*, $z = 2$ is purely real and $z = 2i = 0 + 2i$ is purely imaginary if its real part is zero.

EQUALITY OF TWO COMPLEX NUMBERS

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal if and only if $a = c$ and $b = d$. However there is no ordered relation in the set of complex numbers. The expression $a + ib < c + id$ and $a + ib > c + id$ are meaningless unless $b = d = 0$.

ALGEBRA OF COMPLEX NUMBERS

Let $z_1 = a + ib$, $z_2 = c + id$, then

- $z_1 + z_2 = (a + c) + i(b + d)$
- $z_1 - z_2 = (a - c) + i(b - d)$
- $z_1 z_2 = (a + ib)\cdot(c + id) = (ac - bd) + i(ad + bc)$
- $\dfrac{z_1}{z_2} = \dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{c^2+d^2} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

Conjugate of a Complex Number $(\bar{z})$

Let $z = a + ib$, then its conjugate is denoted by $\bar{z}$ and written by $\bar{z} = a - ib$. Two complex numbers are said to be conjugate of each other if they differ in sign in their imaginary parts only *i.e.*, $a - ib$ and $a + ib$ are said to be conjugate of each other.

- If $z = a + ib$, then $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- $\overline{z_1 z_2} = (\overline{z_1})(\overline{z_2}), \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $(z_2 \neq 0)$

Integral Powers of i

- $(i)^0 = 1, (i)^1 = i, (i)^2 = i^2 = -1, i^3 = i^2 \cdot i = -i,$
  $i^4 = (i^2)^2 = (-1)^2 = 1$
- $i^{4m} = 1$ $(m \in N)$, $i^{4m+1} = (i^{4m})i = i$, $i^{4m+2} = i^2 = -1$,
  $i^{4m+3} = -i$

$\therefore$ $i^{4m+r} = (i^{4m})(i)^r = (1)^m i^r$

GEOMETRIC REPRESENTATION OF COMPLEX NUMBERS

There is one to one correspondence between the set of complex numbers and the point in xy-plane.

$z = a + ib$ corresponds to the point (a, b)

$\bar{z} = a - ib$ corresponds to the point $(a, -b)$

$-z = -a - ib$ corresponds to the point $(-a, -b)$

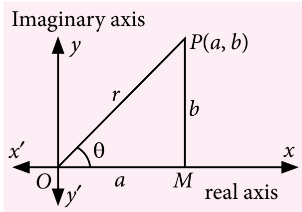
$-\bar{z} = -a + ib$ corresponds to the point $(-a, b)$

The number corresponds to origin is $(0, 0)$.

The real number corresponds to the x-axis called real axis. The purely imaginary number corresponds to the y-axis called imaginary axis. The x-y or xy plane is called complex or Argand plane.

Note: To each complex number there corresponds one & only one point in the plane & conversely to each point in the plane there corresponds one and only one complex number as the point z.

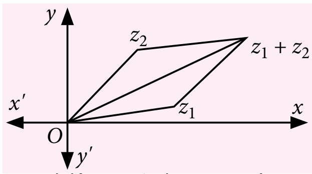
VECTOR REPRESENTATION OF COMPLEX NUMBERS



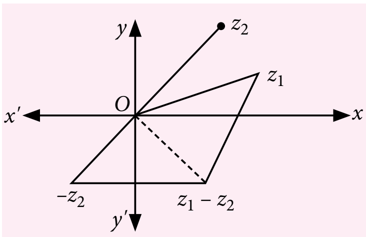
Every complex number can be considered as the position vector of a point. If O is origin and $P(a, b)$ be a point in the argand plane corresponding to the complex number $z = a + ib$, then

- $\overrightarrow{OP} = a\hat{i} + b\hat{j} \Rightarrow |\overrightarrow{OP}| = \sqrt{a^2 + b^2} = |z| = r$
  (Modulus of z represented by r)
- $\arg z$ = direction vector $\overrightarrow{OP} = \tan^{-1}\left(\dfrac{b}{a}\right)$, where $\arg$ of z is denoted by θ.

Representation of $z_1 \pm z_2$



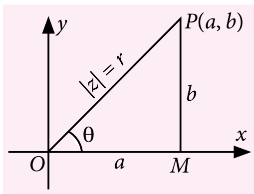
Consider the points z_1 and z_2. Complete the parallelogram with Oz_1 and Oz_2 as adjacent sides where 'O' is origin. The fourth point is $z_1 + z_2$. Similarly, we get the point $z_1 - z_2$ from the point z_1 and $-z_2$.



EULER AND POLAR FORM OF A COMPLEX NUMBER

For a complex number $z = a + ib$ $(a, b \in R)$ let $a = r\cos\theta$, $b = r\sin\theta$, then $z = r(\cos\theta + i\sin\theta)$ is called polar form or trigonometrical form.

If $z = a + ib$, then $r = |z| = \sqrt{a^2 + b^2}$ is modulus of z and $\theta = \text{Arg } z = \tan^{-1}\left(\dfrac{b}{a}\right)$.



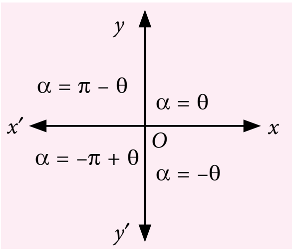
Arg of z is inclination of OP (O is origin) with the positive x-axis.

Also, $z = a + ib = r(\cos\theta + i\sin\theta)$
(polar form) $= r\,\text{cis}\theta$
$= re^{i\theta}$ (Euler form of z)

$\therefore \quad \bar{z} = re^{-i\theta}, \ \frac{1}{z} = \frac{1}{r}e^{-i\theta}$ and $\frac{1}{\bar{z}} = \frac{1}{r}e^{i\theta}$.

Notes:

- If θ be a argument of a complex number then $2n\pi + \theta$, $n \in I$ is also an argument of z.
- If $z = a + ib$ and $P(a, b)$ is a point corresponds to z, α = general Arg of z.
- If $P(a, b)$ lies in the first quadrant *i.e.*, $a > 0$, $b > 0$, then $\theta = \alpha$.
- If $P(a, b)$ lies in the second quadrant *i.e.*, $a < 0$, $b > 0$, then $\alpha = \pi - \theta$.



- If $P(a, b)$ lies in the third quadrant *i.e.*, $a < 0$, $b < 0$, then $\alpha = -\pi + \theta$.
- If $P(a, b)$ lies in the fourth quadrant *i.e.*, $a > 0$, $b < 0$, then $\alpha = -\theta$.
- If $P(a, b)$ lies on positive direction of x-axis *i.e.*, $y = 0$, then $\alpha = 0$ and if lies on negative direction of x-axis, then $\alpha = \pi$.
- If $P(a, b)$ lies on positive direction of y-axis *i.e.*, $x = 0$, then $\alpha = \pi/2$ and if lies on negative direction of y-axis, then $\alpha = -\pi/2$.

MORE FACTS ON ARGUMENT OF COMPLEX NUMBERS

- $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$
- $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$
- $\text{Arg}(z_1\bar{z}_2) = \text{Arg}\, z_1 - \text{Arg}\, z_2 = \text{Arg}\left(\frac{z_1}{z_2}\right)$
- $\text{Arg}(\bar{z}) = -\text{Arg}\, z$
- $\text{Arg}(z_1z_2 \ldots z_n) = \text{Arg}(z_1) + \text{Arg}(z_2) + \ldots + \text{Arg}(z_n)$
- $\text{Arg}\left(\frac{z}{\bar{z}}\right) = 2\,\text{Arg}\, z$ and $\text{Arg}(z^k) = k\,\text{Arg}\, z + 2n\pi, n \in I$
- Arg of $z = 0$ is not-defined.

PROPERTIES OF MODULUS OF COMPLEX NUMBERS

- $|z|^2 = z\bar{z}$, $|z| = |-z| = |\bar{z}| = |-\bar{z}|$
- $-|z| \le \text{Re}(z) \le |z|$, $-|z| \le \text{Im}(z) \le |z|$
- $|z| \le |\text{Re}\, z| + |\text{Im}\, z| \le \sqrt{2}\,|z|$
- $|z_1 z_2| = |z_1||z_2|$, $|z_1 z_2 z_3 \ldots z_n| = \prod_{i=1}^{n}|z_i| = |z_1|\,|z_2| \ldots |z_n|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \ (z_2 \ne 0)$

TRIANGLE INEQUALITY

- $||z_1| - |z_2|| \le |z_1 \pm z_2| \le |z_1| + |z_2|$
- $|z_1 + z_2| = |z_1| + |z_2| \Rightarrow \text{Arg}\, z_1 - \text{Arg}\, z_2 = 0, \ \frac{z_1}{z_2} > 0$ and $0, z_1, z_2$ are collinear such that z_1, z_2 lies on the same side of origin.
- $|z_1 - z_2| = |z_1| + |z_2| \Rightarrow \text{Arg}\, z_1 - \text{Arg}\, z_2 = \pi, \ \frac{z_1}{z_2} < 0$ and z_1, z_2 are collinear such that $z_1 < 0 < z_2$.
- $|z_1 - z_2| = |z_1 + z_2| \Rightarrow \text{Arg}\, z_1 - \text{Arg}\, z_2 = \pm\frac{\pi}{2}$, $\bar{z}_1 z_2$ and $\frac{z_1}{z_2}$ are purely imaginary numbers.
- **Parallelogram Law :** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
- **De-Moivre's Theorem :** $(e^{i\theta})^n = e^{in\theta}$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is any rational number.

Note : (i) De-Moivre's theorem is not applicable on $z = \sin\theta + i\cos\theta$.

(ii) De-Moivre's theorem is use to develop the various trigonometrical formula's like $\sin 2\theta$, $\sin 3\theta$, …, $\sin n\theta$, $\cos\theta$, $\cos 2\theta$, …, $\cos n\theta$, $\tan\theta$, $\tan 2\theta$, …, $\tan n\theta$ and so on.

CUBE ROOTS OF UNITY

If $z = (1)^{1/3} \Rightarrow z^3 - 1 = 0$, then the roots are $1, \omega, \omega^2$, where $\omega = \frac{-1+i\sqrt{3}}{2}$ then $\omega^2 = -\frac{1-i\sqrt{3}}{2}$ and euler form of $x = 1, \omega, \omega^2 = 1, e^{2i\pi/3}, e^{i4\pi/3}$.

- $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$, $\bar{\omega} = \frac{1}{\omega} = \omega^2$, $\bar{\omega}^2 = \frac{1}{\omega^2} = \omega$
- $\omega^n = \omega^r$, where 'r' is the remainder when the integer (n) is divided by 3 *i.e.*, $\omega^{3q+r} = \omega^r \cdot \omega^{3q} = \omega^r$ (as $\omega^3 = 1$)
- $1 + \omega^r + \omega^{2r} = \begin{cases} 0, & \text{if } r \text{ is not a multiple of 3} \\ 3, & \text{if } r \text{ is a multiple of 3} \end{cases}$
- $1 \cdot \omega \cdot \omega^2 = 1 \cdot e^{\frac{2\pi i}{3}} \cdot e^{\frac{4i\pi}{3}} = e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1$
- ω and ω^2 are conjugate, reciprocal and square of each other.
- $1, \omega, \omega^2$ form an equilateral triangle inscribed in the circle of unit radius *i.e.* $|z| = 1$

Note : 1. Cube roots of -1 are $-1, -\omega, -\omega^2$

2. Cube roots of 8 are $2, 2\omega, 2\omega^2$

3. Cube roots of -8 are $-2, -2\omega, -2\omega^2$.

USE OF CUBE ROOTS OF UNITY IN FACTORIZATION

- $x^2 \pm x + 1 = (x \mp \omega)(x \mp \omega^2)$
- $x^2 \pm xy + y^2 = (x \mp \omega y)(x \mp \omega^2 y)$

- $x^3 \pm y^3 = (x \pm y)(x \pm \omega y)(x \pm \omega^2 y)$
- $x^2 + y^2 + z^2 - xy - yz - xz = (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$
- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$
- $x^2 + y^2 = (x + iy)(x - iy)$

PROBLEMS

Single Correct Answer Type

1. If $f(x) \cdot f(y) = f(x) + yf(y)$, where $f(y) \neq 0\ \forall\ y \in N$, then $fofofofo \ldots of(1)$ upto 2017 times equals to
(a) 2018 (b) 2017
(c) 2016 (d) None of these

2. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be such that $a, b, c, d \in R$, $f(2i) = f(2 + i) = 0$. If the product of a, b, c, d be expressed in terms of its prime factors, then the product of the exponents of prime factor is
(a) 64 (b) 32
(c) 16 (d) None of these

3. If S is the sum of the product of roots of the equation $x^{2018} - \left(\frac{1}{2} - x\right)^{2018} = 0$ taken two at a time, the tens place of the sum of the digits of S equals to
(a) 3 (b) 2 (c) 1 (d) 0

4. A function f is defined for all positive integers and satisfies $f(1) = 4035$ and $f(1) + f(2) + \ldots + f(n) = n^2 f(n)$, then the value of $\frac{1}{f(4035)}$ equal; $\forall\ n > 1$
(a) 2015 (b) 2016 (c) 2017 (d) 2018

5. If $a + b + c = 0$, $a^3 + b^3 + c^3 = 12$ and $a^5 + b^5 + c^5 = 40$, then the value of $a^4 + b^4 + c^4$ equals to
(a) 8 (b) 6 (c) 4 (d) 2

6. The real value of 'a' for which sum of the square of the roots of the equation $x^2 - (a - 3)x - (a + 1) = 0$ assume the least value is
(a) 4 (b) 3 (c) 1 (d) 2

7. If the equation $(x + 1)^n + x^n + 1 = 0$ has $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ as its roots and α and β satisfies both equation $x^2 + px + q = 0$ and $x^{4036} + p^{2018}x^{2018} + q^{2018} = 0$, then the value of n must be
(a) 2019 (b) 2018 (c) 2017 (d) 2016

8. If α, β are roots of $ax^2 + 2bx + c = 0$ and $\alpha - h$, $\beta - h$ are roots of $Ax^2 + 2Bx + k = 0$ and D_1, D_2 are their discriminants respectively, then ratio of $D_1 : D_2$ is
(a) $2a : 3B$ (b) $3A : 2b$ (c) $a : A^2$ (d) $a^2 : A^2$

9. If α, β are roots of the equation $x^2 - x + 1 = 0$, then value of $\alpha^{2018} + \beta^{2018}$ equals to
(a) 0 (b) 1
(c) -1 (d) None of these

10. If $z_n = \text{cis}\left(\frac{\pi}{n(n+1)(n+2)}\right)$ for $n = 1, 2, 3 \ldots$ and $z = \lim_{n\to\infty} (z_1 z_2 \ldots z_n)$, then the principal argument of z is
(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) $2\pi/3$

More Than One Correct Answer Type

11. If roots of the equation $z^4 + az^3 + (-36 + 15i)z^2 + bz = 0$ are vertices of a square, then $(a + b)$ can be equal to
(a) $39 + 65i$ (b) $-39 - 65i$
(c) $35 + 45i$ (d) $-35 - 45i$

12. If $z_1 = a + ib$ and $z_2 = c + id$ (where $i = \sqrt{-1}$) are two complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies,
(a) $|w_1| = 1$ (b) $|w_2| = 1$
(c) $|\bar{w}_1 w_2| = 1$ (d) $\text{Re}(w_1 \bar{w}_2) = 0$

13. If $z^3 + (3 + 2i)z + (-1 + ai) = 0$ has one real root, then the value of 'a' lies in the interval, (a is real number)
(a) $(0, 1)$ (b) $(-1, 0)$ (c) $(-2, 3)$ (d) $(-2, 1)$

14. If the roots of equation $x^3 + bx^2 + cx - 1 = 0$ form an increasing G.P., then
(a) one of the root is 1 (b) $b \in (-\infty, -3)$
(c) $b + c = 0$
(d) one root is <1 and other root is > 1

15. If a, b, c are positive rational numbers satisfies $a > b > c$ and the quadratic equation $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ has a root in the interval $(-1, 0)$, then
(a) both roots of the equation are rational
(b) $c + a < 2b$
(c) the equation $ax^2 + 2bx + c = 0$ has both negative real roots.
(d) the equation $cx^2 + 2ax + b = 0$ has both negative real roots.

16. Which of the following is true if $z = x + iy$
(a) The equations $|z - 2| = 1$ and $|z - 1| = 2$ have only one solution.
(b) The equation $z^2 + 8\bar{z} = 0$ has 4 solutions.
(c) The number of solutions of $z^2 + \bar{z}^2 = 0$ are infinite.
(d) The equation $z^2 + |z| = 0$ have more than two solutions.

17. If $x^2 - ix + 1 = 0$, then $x^{10} + \frac{1}{x^{10}}$ is an integer divisible by
(a) 3 (b) 41 (c) 11 (d) 13

18. If $(x + 1)^4 + (x + 3)^4 = 16$ then value/values of x are
(a) -1 (b) -3
(c) $-2 - i\sqrt{7}$ (d) $-2 + i\sqrt{7}$

19. The product of real roots of the equation $|x|^{\frac{4}{5}} - |x|^{\frac{2}{5}} - 12 = 0$ is
(a) 1024 (b) −1024 (c) 4^5 (d) -2^{10}

20. If $c_0, c_1, c_2, ..., c_n$ are binomial coefficients in the expansion of $(1 + x)^n$, then which of the following is true/correct?

(a) $c_0 - c_2 + c_4 - c_6 + \ldots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$ and
$c_1 - c_3 + c_5 - c_7 + \ldots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$

(b) $c_0 + c_4 + c_8 \ldots = 2^{n-2} + 2^{\frac{n-2}{2}}\cos\frac{n\pi}{4}$ and
$c_2 + c_6 + c_{10} + \ldots = 2^{n-2} - 2^{\frac{n-2}{2}}\cdot\cos\frac{n\pi}{4}$

(c) $c_1 + c_5 + c_9 + \ldots = 2^{n-2} + 2^{\frac{n-2}{2}}\cdot\sin\frac{n\pi}{4}$ and
$c_3 + c_7 + c_{11} + \ldots = 2^{n-2} - 2^{\frac{n-2}{2}}\cdot\sin\frac{n\pi}{4}$

(d) $c_0 + c_3 + c_6 + \ldots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$ and
$c_1 + c_4 + c_7 + \ldots = \frac{1}{3}\left(2^n - \cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}\right)$

Comprehension Type

Paragraph for Q. No. 21 to 24

Let $(a+\sqrt{b})^{f(x)} + (a-\sqrt{b})^{f(x)+\lambda m} = A$, where $\lambda m = N$, $A \in R$ and $a^2 - b = 1$

$\therefore$ $(a+\sqrt{b})(a-\sqrt{b}) = 1 \Rightarrow (a+\sqrt{b}) = (a-\sqrt{b})^{-1}$ and $(a-\sqrt{b}) = (a+\sqrt{b})^{-1}$ which means $P^{f(x)} + Q^{f(x)} = P + Q$ $\Rightarrow f(x) = \pm 1$. On the basis of above information answer the following questions.

21. The solution set of the equation
$(7+4\sqrt{3})^{x^2+2x+1} + (7-4\sqrt{3})^{x^2+2x-1} = \frac{14}{7-4\sqrt{3}}$ is
(a) $\sqrt{2}-1$ (b) −1
(c) $-(\sqrt{2}+1)$ (d) All of these

22. If $(\sqrt{7+4\sqrt{3}})^{\sqrt{a\sqrt{a\sqrt{a\ldots\infty}}}} + (2-\sqrt{3})^{x^2+x-5-\sqrt{x\sqrt{x\sqrt{x\ldots\infty}}}} = 4$, where $a = x^2 - 5$, then x equals to
(a) $\sqrt{3}$ (b) $\sqrt{5}$ (c) $\sqrt{6}$ (d) $-\sqrt{6}$

23. If α, β are roots of the equation $x^2 - 8x + 1 = 0$ where α > β, then number of solutions of the equation
$\alpha^{y^2-2y+1} + \beta^{y^2-2y-1} = \frac{50}{7(4-\sqrt{15})}$ is/are
(a) 4 (b) 3 (c) 2 (d) 1

24. The number of real roots of the equation $(17+12\sqrt{2})^t + (17-12\sqrt{2})^t = 34$ (where $t = x^2 - 2|x|$) are
(a) 1 (b) 2 (c) 3 (d) 4

Matrix Match Type

25. Match the following :

	Column-I		Column-II
P.	If $x + \frac{1}{x} = \sqrt{3}$, then $x^{2018} + \frac{1}{x^{2018}}$ equals to	1.	2
Q.	Let $z = 1 + i\sqrt{3}$, then $\frac{z^6 + (\bar{z})^6}{2^5}$ equals to	2.	1
R.	If 1, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n n^{th} roots of unity. If n is an odd natural number, then the value of $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots(1+\alpha_{n-1})$ equals to	3.	3
S.	If $z - \frac{1}{z} = i$, then $z^{2018} + \frac{1}{z^{2018}}$ equals to	4.	4
T.	If $\lvert z + 4i\rvert \le 2$, then minimum value of $\lvert z + 3\rvert$ =		
U.	If $x = -2 - i\sqrt{3}$, then value of $2x^4 + 5x^3 + 7x^2 - x + 37$ =		

Numerical Answer Type

26. If ω ≠ 1 is a cube root of unity and $a - b = 41$ and $a^3 - b^3 = 4223$, then value of $(a\omega^2 - b\omega)(a\omega - b\omega^2)$ equals to

27. If $2^8\cos^4\theta\sin^5\theta = a\sin 9\theta + b\sin 7\theta - c\sin 5\theta + d\sin 3\theta + e\sin\theta$, where θ is real and value of $a^2 + b^2 + c^2 + d^2 + e^2$ equals 10λ, then λ equals to

28. Let a, b, c are distinct integers such that $c = b + 1$ and $b = a + 1$ and ω is the cube root of unity other than 1. If minimum value of $|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$ is $\alpha^{1/6}$, then α equals to

29. The number of solutions of the equation $2^x + 3^x + 6^x - 4^x - 9^x = 1$ is/are

30. Number of common roots of the equations $z^3 - (1 + i)z^2 + (1 + i)z - i = 0$ (where $i = \sqrt{-1}$) and $z^{2017} + z^{2018} - 1$ is/are

SOLUTIONS

1. (a) : Since $f(y) \ne 0 \Rightarrow f(x) \ne 0$
$\Rightarrow$ $f(1), f(2), f(3), \ldots, f(n)$ are all non-zero quantities.
Given, $f(x)\cdot f(y) = f(x) + yf(y)$ ...(*)
$\Rightarrow f(1)\cdot f(1) = f(1) + f(1)$ [Putting $x = y = 1$]
$\Rightarrow f(1)f(1) = 2f(1) \Rightarrow f(1) = 2$ as $f(1) \ne 0$
Again, $f(2)f(2) = f(2) + 2f(2)$ [Putting $x = y = 2$ in (*)]
$\Rightarrow f(2) = 3$

Similarly continue the process n times, we get general result $f(n) = n + 1$

$\therefore$ $fofofo \ldots of(1)$ (upto 2017 times)

$= fofofo \ldots of(2)$ (upto 2016 times)

$= f(2017) = 2018$

2. **(c) :** $\because f(2i) = f(2 + i) = 0$

$\Rightarrow$ $x = 2i, -2i, 2 + i$ and $2 - i$ are zeros of $f(x)$

$\therefore$ $f(x) = x^4 + ax^3 + bx^2 + cx + d$

$= (x + 2i)(x - 2i)(x - 2 + i)(x - 2 - i)$

$\Rightarrow$ $x^4 + ax^3 + bx^2 + cx + d = (x^2 + 4)[(x - 2)^2 + 1]$

$= (x^2 + 4)(x^2 - 4x + 5)$

$\Rightarrow$ $x^4 + ax^3 + bx^2 + cx + d = x^4 - 4x^3 + 9x^2 - 16x + 20$

$\Rightarrow$ $abcd = (-4)(9)(-16)(20) = 2^8 \cdot 3^2 \cdot 5^1$

$\therefore$ Product of exponents of prime factors $= 8 \times 2 \times 1$

$= 16$

3. **(a) :** The given equation $x^{2018} - \left(\frac{1}{2} - x\right)^{2018} = 0$

$$\Rightarrow x^{2018} - \left(x - \frac{1}{2}\right)^{2018} = 0$$

$$\Rightarrow \binom{2018}{1} x^{2017}\left(\frac{1}{2}\right) - \binom{2018}{2}\left(\frac{1}{2}\right)^2 x^{2016} + \binom{2018}{3}\left(\frac{1}{2}\right)^3 x^{2015} + \ldots - \left(\frac{1}{2}\right)^{2018} = 0$$

$\Rightarrow$ S = Sum of the product of roots taken two at a time

$$= \frac{\text{Coefficient of } x^{2015}}{\text{Coefficient of } x^{2017}}$$

$$= \frac{\binom{2018}{3}\left(\frac{1}{8}\right)}{\binom{2018}{1}\left(\frac{1}{2}\right)} = \frac{2018 \times 2017 \times 2016}{3 \times 2 \times 1} \times \frac{1}{4} \times \frac{1}{2018}$$

$= 2017 \times 84 = 169428$

$\therefore$ Sum of the digits of $S = 1 + 6 + 9 + 4 + 2 + 8 = 30$

$\Rightarrow$ Tens place of the sum of the digits of S is 3.

4. **(d) :** Given, $n^2 f(n) = f(1) + f(2) + f(3) + \ldots + f(n)$ ...(i)

Since $n > 1$

$\therefore$ Taking $n = 2$ in (i), we have

$$2^2 f(2) = f(1) + f(2) \Rightarrow 3f(2) = f(1) \Rightarrow f(2) = \frac{f(1)}{1+2}$$

Again taking $n = 3$ in (i), we have

$9f(3) = f(1) + f(2) + f(3)$

$$\Rightarrow 8f(3) = f(1) + f(2) = \frac{4}{3} f(1)$$

$$\Rightarrow f(3) = \frac{f(1)}{6} = \frac{f(1)}{1+2+3}$$

Continuing the process like this, we have

$$f(4035) = \frac{f(1)}{1+2+3+\ldots+4035} = \frac{2f(1)}{4035 \times 4036} = \frac{f(1)}{4035 \times 2018}$$

$$\Rightarrow f(4035) = \frac{1}{2018} \qquad [\because f(1) = 4035 \text{ (Given)}]$$

$$\Rightarrow \frac{1}{f(4035)} = 2018$$

5. **(a) :** As $a + b + c = 0$, we say that a, b, c are roots of $x^3 + px + q = 0$ ...(i)

$\therefore$ $ab + bc + ca$ = Sum of the product of roots of (i) taken two at a time

$\Rightarrow$ $ab + bc + ca = p$ and $abc = -q$

Now, $a + b + c = 0 = \sum a$

$\Rightarrow$ $a^3 + b^3 + c^3 = 3abc$

$$\Rightarrow abc = \frac{a^3 + b^3 + c^3}{3} = \frac{12}{3} = 4$$

$[\because a^3 + b^3 + c^3 = 12 \text{ (Given)}]$

Since, $abc = -q \Rightarrow q = -4$

Also, $a + b + c = 0 \Rightarrow a^2 + b^2 + c^2 = -2(ab + bc + ca)$

$\Rightarrow$ $a^2 + b^2 + c^2 = -2(p)$

$\because$ a, b, c are roots of $x^3 + px + q = 0$

$\Rightarrow$ $a^3 + pa + q = 0 \Rightarrow a^5 + pa^3 + qa^2 = 0$

$\Rightarrow$ $\sum a^5 + p\sum a^3 + q\sum a^2 = 0$

$\Rightarrow$ $40 + p(12) + q(-2p) = 0 \Rightarrow p = -2$ $(\because q = -4)$

$\therefore$ The cubic equation (i) reduces to $x^3 - 2x - 4 = 0$

$\Rightarrow$ $a^3 - 2a - 4 = 0$ $[\because a$ is root of (i)$]$

$\Rightarrow$ $a^4 - 2a^2 - 4a = 0$

$\Rightarrow$ $\sum a^4 = 2\sum a^2 + 4\sum a = 2 \times (-2p) + 4(0) = -4p$

$= (-4)(-2) = 8$ $\therefore$ $a^4 + b^4 + c^4 = 8$

6. **(d) :** The given equation is

$x^2 - (a - 3)x - (a + 1) = 0$ ...(i)

Now, $D = b^2 - 4ac = (a - 3)^2 + 4(a + 1) = a^2 - 2a + 13$

$= (a - 1)^2 + 12 > 0 \ \forall$ values of 'a'.

$\Rightarrow$ Roots of the equation are real & distinct. Let roots are α and β.

$\therefore$ $\alpha + \beta = a - 3, \alpha\beta = -(a + 1)$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (a - 3)^2 + 2(a + 1) = (a - 2)^2 + 7$

For least value of $\alpha^2 + \beta^2$, we must have $(a - 2)^2 = 0$

i.e., $a = 2$.

7. **(b) :** Given $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are roots of $(x + 1)^n + x^n + 1 = 0$

$$\Rightarrow \left(\frac{\alpha}{\beta} + 1\right)^n + \left(\frac{\alpha}{\beta}\right)^n + 1 = 0$$

$\Rightarrow$ $(\alpha + \beta)^n + \alpha^n + \beta^n = 0$

$\Rightarrow$ $\alpha^n + \beta^n + (-p)^n = 0$ ...(A)

$[\because \alpha, \beta$ are roots of $x^2 + px + q = 0)$

$\Rightarrow \alpha + \beta = -p$ and $\alpha\beta = q]$

Again α, β are roots of $x^{4036} + p^{2018}x^{2018} + q^{2018} = 0$

$\therefore \quad \alpha^{4036} + p^{2018}\alpha^{2018} + q^{2018} = 0$ ...(i)

and $\beta^{4036} + p^{2018}\beta^{2018} + q^{2018} = 0$ ...(ii)

Subtracting (ii) from (i), we get

$\alpha^{4036} - \beta^{4036} + p^{2018}(\alpha^{2018} - \beta^{2018}) = 0$

$\Rightarrow \quad (\alpha^{2018})^2 - (\beta^{2018})^2 + p^{2018}(\alpha^{2018} - \beta^{2018}) = 0$

$\Rightarrow \quad (\alpha^{2018} - \beta^{2018})(\alpha^{2018} + \beta^{2018} + p^{2018}) = 0$

$\Rightarrow \quad \alpha^{2018} + \beta^{2018} + p^{2018} = 0 \quad (\because \ \alpha^{2018} \neq \beta^{2018}$ as $\alpha \neq \beta)$

$\Rightarrow \quad \alpha^{2018} + \beta^{2018} + (-p)^{2018} = 0$ ...(B)

From (A) & (B), we get $n = 2018$

8. (d) : Given α, β are roots of $ax^2 + 2bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\dfrac{2b}{a}$ and $\alpha\beta = \dfrac{c}{a}$ ...(i)

Also, $\alpha - h$, $\beta - h$ are roots of $Ax^2 + 2Bx + k = 0$

$\Rightarrow \quad (\alpha - h) + (\beta - h) = -\dfrac{2B}{A}$ and $(\alpha - h)(\beta - h) = \dfrac{k}{A}$ ...(ii)

Why it's not a good idea to be a civil engineer in India

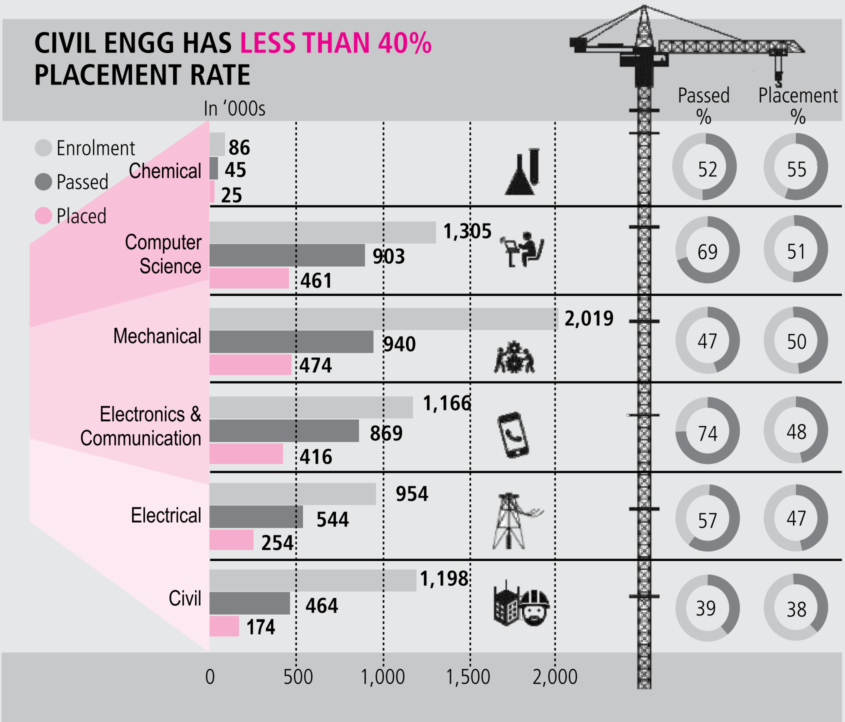
Civil engineering has registered the lowest placement rate among six engineering streams approved by the AICTE.

Among engineering streams, chemical engineering, computer science and mechanical engineering have the highest placement rates.

Those graduating out of engineering colleges do not always possess skills that industry requires resulting in their not getting hired.

Civil engineering has registered the lowest placement rate of a mere 38% between 2012-13 and 2015-16 among six engineering streams approved by the All India Council for Technical Education (AICTE). At a time when the construction sector is one of the fastest growing, this apparent lack of demand for civil engineers comes as quite a surprise. Data from the AICTE shows that over this period chemical engineering had the highest placement rates. Surprisingly, the electronics and communication stream too had a relatively low placement rate of 48%. Only three streams, chemical engineering, computer science and mechanical engineering saw more than half their graduates getting place.

The data also shows a considerable variation in pass percentages across streams, with civil engineering once again having the lowest rate of 39% and electronics and communications registering a 74% success rate in clearing the course. Mechanical engineering continues to be the most popular engineering stream with over 20 lakh students enrolling for the courses approved by the AICTE followed by computer science, civil engineering, electronics and communication, electrical engineering and chemical engineering.



Mechanical engineering has recorded 47% pass outs in these four years, but placement in the stream is barely 50%. Chemical engineering is the least preferred branch with just 86,000 students enrolling in four years. Interestingly, between 2013-14 and 2017-18 only 55% of AICTE approved engineering seats were filled and 214 institutes were closed during this period. Over 77 lakh students enrolled for various engineering streams during this period, over three-fourths of them boys.

According to industry experts, the percentage of engineering seats being filled has come down over time because of an explosion in the number of seats. Another factor, they suggest, is that those graduating out of engineering colleges do not always possess skills that industry requires resulting in, their not getting hired.

Courtesy : The Times of India

Now, $\alpha - \beta = (\alpha - h) - (\beta - h)$

$\Rightarrow \quad (\alpha - \beta)^2 = [(\alpha - h) - (\beta - h)]^2$

$\Rightarrow \quad (\alpha+\beta)^2 - 4\alpha\beta = [(\alpha-h)+(\beta-h)]^2 - 4(\alpha-h)(\beta-h)$

$\Rightarrow \quad \dfrac{4b^2}{a^2} - \dfrac{4c}{a} = \dfrac{4B^2}{A^2} - \dfrac{4k}{A}$ [From (i) and (ii)]

$\Rightarrow \quad \dfrac{4b^2 - 4ac}{a^2} = \dfrac{4B^2 - 4kA}{A^2}$

$\Rightarrow \quad \dfrac{D_1}{a^2} = \dfrac{D_2}{A^2} \Rightarrow \dfrac{D_1}{D_2} = \dfrac{a^2}{A^2} \Rightarrow D_1 : D_2 = a^2 : A^2$

9. (c) : Given, $x^2 - x + 1 = 0 \Rightarrow \left(x - \dfrac{1}{2}\right)^2 = -1 + \dfrac{1}{4} = \dfrac{-3}{4}$

$\therefore \quad \alpha = \dfrac{1}{2} + \dfrac{i\sqrt{3}}{2},\ \beta = \dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}$

$\Rightarrow \quad \alpha = -\left(-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right),\ \beta = -\left(-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right)$

$\Rightarrow \quad \alpha = -\omega^2,\ \beta = -\omega$

$\therefore \quad \alpha^{2018} + \beta^{2018} = (-\omega^2)^{2018} + (-\omega)^{2018}$

$= \omega^{4036} + \omega^{2018} = \omega + \omega^2 = -1$

10. (a) **11. (a, b, c, d)**

12. (a, b, c, d) : Given $|z_1| = |z_2| = 1$

We have, $z_1 = \cos\theta_1 + i\sin\theta_1$ and $z_2 = \cos\theta_2 + i\sin\theta_2$ where θ_1 and θ_2 are arguments of z_1 and z_2 respectively.

Also, $z_1 = a + ib = \cos\theta_1 + i\sin\theta_1$ and

$z_2 = c + id = \cos\theta_2 + i\sin\theta_2$

$\Rightarrow \quad a = \cos\theta_1,\ b = \sin\theta_1$ and $c = \cos\theta_2,\ d = \sin\theta_2$

Again $\text{Re}(z_1\bar{z}_2) = 0 + 0i$

$\Rightarrow \quad \text{Re}[(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)] = 0 + 0i$

$\Rightarrow \quad \text{Re}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = 0 + 0i$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 0 \Rightarrow \theta_1 - \theta_2 = \dfrac{\pi}{2} \Rightarrow \theta_2 = \theta_1 - \dfrac{\pi}{2}$

Now, $w_1 = a + ic$ and $w_2 = b + id$

$\Rightarrow \quad w_1 = \cos\theta_1 + i\cos\theta_2$ and $w_2 = \sin\theta_1 + i\sin\theta_2$

$\Rightarrow \quad w_1 = \cos\theta_1 + i\sin\theta_1 = z_1$

and $w_2 = \cos\theta_2 + i\sin\theta_2 = z_2$

$\Rightarrow \quad |w_1| = |z_1| = 1$ and $|w_2| = |z_2| = 1$

Again, $|\bar{w}_1 w_2| = |\bar{w}_1||w_2| = |w_1||\bar{w}_2| = |w_1||w_2| = 1$

and $\text{Re}(w_1\bar{w}_2) = \text{Re}[(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)]$

$= \text{Re}[\cos(\theta_1 - \theta_2) + i(\sin(\theta_1 - \theta_2))]$

$= \text{Re}\left[\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right] = \text{Re}[(0 + i)]$

$\therefore \quad \text{Re}(w_1\bar{w}_2) = 0$

13. (b, c, d)

14. (a, b, c, d) : Suppose the roots are $\dfrac{\alpha}{r}, \alpha, \alpha r$ where $\alpha > 0$ and $r > 1$.

$\therefore \quad \dfrac{\alpha}{r} + \alpha + \alpha r = -b$ (sum of roots) ...(i)

Sum of the product of roots taken two at a time

$= \alpha^2\left(\dfrac{1}{r} + r + 1\right) = c$ ...(ii)

Product of roots taken all at a time $= \dfrac{\alpha}{r}(\alpha)(\alpha r) = 1$ ...(iii)

$\Rightarrow \quad \alpha^3 = 1 \Rightarrow \alpha = 1$ (other values are imaginary)

$\therefore \quad$ Sum of roots $= \dfrac{1}{r} + 1 + r = -b$

$\Rightarrow \quad \left(\dfrac{1}{\sqrt{r}} - \sqrt{r}\right)^2 + 3 = -b$

$\Rightarrow \quad \left(\dfrac{1}{\sqrt{r}} - \sqrt{r}\right)^2 = -(b + 3) > 0$

$\Rightarrow \quad b < -3 \Rightarrow b \in (-\infty, -3)$

Again from sum of the product of roots taken two at a time and using $\alpha = 1$, we have $\dfrac{1}{r} + r + 1 = c$ and $\dfrac{1}{r} + r + 1 = -b$

SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique solution.

6				9	8	3		7				4			1	5				6
			6		1									9	6		3			
			2					4							8					
	6		1		7	8								1	7		8	5	9	
7				4			6					9			5	1				3
1	8	5	9	3								5					9		6	7
	2		3							5	2						1			8
							4				7		6			8				
9				8	4				9		4							4		9
						6	3	7				5								
						4								9						
								8				7	4	2						
1		8							2		1				2	1				3
				3			1		7				2							
9			7						8	4							4		6	
5	6		4					9								2	5	7	3	4
8				9	3			6					8			7				1
	1	9	8		5	2								2	1		9		8	
					1							1					2			
			6		7	4									6		3			
2				4	9			7				3		8	4	5				2

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

$\Rightarrow \quad b = -c \Rightarrow b + c = 0$

Now, $r > 1$ and roots are $\frac{\alpha}{r}$, α, αr

$\Rightarrow \quad \frac{\alpha}{r} = \frac{1}{r} < 1$ and $\alpha r = r > 1$

$\therefore$ One root is < 1 and other root is > 1.

15. (a, b, c, d) : Since, we know that if sum of the coefficients of a quadratic equation is zero, then roots of the equation be rational and one of the root be 1 and other root be $\frac{\text{constant term}}{\text{coefficient of } x^2}$.

Here, sum of the coefficients is $(a + b - 2c) + (b + c - 2a) + (c + a - 2b) = 0$.

$\therefore$ One of the root is 1 and other root is $\frac{c+a-2b}{a+b-2c}$ which is rational as a, b, c are rational. Thus both roots of the equation are rational.

Also, $a > b > c \Rightarrow a - b > 0, a - c > 0, b - c > 0$ ...(i)

Again, root of the equation lies in the interval $(-1, 0)$

$\therefore$ $f(-1) f(0) < 0$, where $f(0) = c + a - 2b$ and $f(-1) = 2(2a - b - c)$

$\Rightarrow \quad 2(2a - b - c)(c + a - 2b) < 0$

$\Rightarrow \quad [(a - c) + (a - b)](c + a - 2b) < 0$

$\Rightarrow \quad (c + a - 2b) < 0$ (using (i))

$\Rightarrow \quad c + a < 2b$ ...(ii)

Now, in order to check that equation $ax^2 + 2bx + c = 0$ has both roots real and negative we need to show $D > 0$.

Sum of roots is < 0, sum is clearly < 0 as $S = -\frac{2b}{a} < 0$ as $a, b > 0$.

Again, $(c + a) < 2b$ [From (ii)]

$\Rightarrow \quad 2b > (c + a)$

$\Rightarrow \quad 4b^2 > (c + a)^2$

$\Rightarrow \quad 4b^2 > c^2 + a^2 + 2ac$

$\Rightarrow \quad 4b^2 - 4ac > c^2 + a^2 - 2ac$

$\Rightarrow \quad 4b^2 - 4ac > (c - a)^2 > 0$

$\Rightarrow \quad D > 0$ and $a, b, c > 0$ and $a > b > c$, $S = \frac{-2b}{a} < 0$

$\therefore$ Roots of the equation $ax^2 + 2bx + c = 0$ are real and negative.

Similarly, discriminant of $cx^2 + 2ax + b = 0$ is $D = 4a^2 - 4bc = 4(a^2 - bc) > 0$ as $a > b > c$, then $a^2 > bc$

$\therefore \quad D > 0$

Thus roots of $cx^2 + 2ax + b = 0$ are real and negative.

16. (a, b, c, d) : Given, $z = x + iy$

(a) $\therefore \quad |z - 2| = 1$ and $|z - 1| = 2$

$\Rightarrow \quad (x - 2)^2 + y^2 - 1 = 0$ and $(x - 1)^2 + y^2 - 4 = 0$

$\Rightarrow \quad x^2 + y^2 - 4x + 3 = 0$ and $x^2 + y^2 - 2x - 3 = 0$

$\Rightarrow \quad x = 3$ and $y = 0$ (Solving above two equations)

$\therefore \quad z = 3 + 0i$, is the only one solution

(b) Given, $z^2 + 8\bar{z} = 0$

$\Rightarrow \quad z^2 = -8\bar{z}$

$\Rightarrow \quad z\bar{z} = -8\bar{z}$

$\Rightarrow \quad |z|\,|z| = 8|z|$ $(\because |z| = |\bar{z}|)$

$\Rightarrow \quad |z| = 0, |z| = 8 \Rightarrow |z|^2 = 64$

$\Rightarrow \quad z\bar{z} = 64 \therefore \bar{z} = \frac{64}{z}$

Now, $z^2 = -8\bar{z}$

$\Rightarrow \quad z^2 = (-8)\left(\frac{64}{z}\right) \Rightarrow z^3 = (-8)^3 \therefore z = -8, -8\omega, -8\omega^2$

Now, $|z| = 0$ gives one solution and $|z| = 8$ gives three solutions.

$\therefore$ Total number of solutions of $z^2 + 8\bar{z} = 0$ is 4.

(c) Given, $z^2 + \bar{z}^2 = 0$. Here $z = x + iy \therefore \bar{z} = x - iy$

$\Rightarrow \quad 2(x^2 - y^2) = 0 \Rightarrow x = \pm y$ or $y = \pm x$

$\Rightarrow \quad z = \pm y + iy$ or $z = x \pm ix$

$\Rightarrow \quad z = y(\pm 1 + i)$ or $z = x(1 \pm i)\ \forall\ x, y \in R$

$\Rightarrow \quad z^2 + \bar{z}^2 = 0$ have infinite solutions as x, y can assume any real number.

(d) Given $z^2 + |z| = 0 \Rightarrow z^2 = -|z|$

(where $z^2 = x^2 - y^2 + 2ixy$)

$\therefore \quad |z^2| = |-|z|| = |z|$

$\Rightarrow \quad |z|^2 - |z| = 0$

$$\left[\begin{array}{l} \because \ |z^2| = |x^2 - y^2 + 2ixy|\sqrt{(x^2-y^2)^2 + 4x^2y^2} \\ \Rightarrow \ |z^2| = \sqrt{(x^2+y^2)^2} = x^2 + y^2 = |z|^2 \end{array}\right]$$

$\Rightarrow \quad |z|(|z| - 1) = 0$

$\Rightarrow \quad |z| = 0, |z| = 1$

$\Rightarrow \quad x = 0, y = 0, z = 0 \pm i$ i.e., $z = \pm i$

$\therefore \quad z = 0, z = \pm i$ i.e., $z = 0, i, -i$

Thus, $z^2 + |z| = 0$ has three solutions.

17. (a, b) : Given $x^2 - ix + 1 = 0 \Rightarrow x + \frac{1}{x} = i$

$\therefore \quad x^2 + \frac{1}{x^2} + 2 = i^2 \Rightarrow x^2 + \frac{1}{x^2} = -3$ ...(A)

Again $x + \frac{1}{x} = i$

$\therefore \quad x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = i^3$

$\Rightarrow \quad x^3 + \frac{1}{x^3} + 3i = -i \Rightarrow x^3 + \frac{1}{x^3} = -4i$ ...(B)

Multiply (A) and (B), we have

$$\left(x^3 + \frac{1}{x^3}\right)\left(x^2 + \frac{1}{x^2}\right) = (-3)(-4i)$$

$$\Rightarrow\ x^5 + \frac{1}{x^5} + \left(x + \frac{1}{x}\right) = 12i$$

$$\Rightarrow\ x^5 + \frac{1}{x^5} = 11i$$

$$\Rightarrow\ x^{10} + \frac{1}{x^{10}} = -121 - 2$$

$$\Rightarrow\ x^{10} + \frac{1}{x^{10}} = -123 = -(3)\ (41)$$

$\Rightarrow\ x^{10} + \frac{1}{x^{10}}$ is divisible by 3 and 41.

18. (a, b, c, d) : The given equation is of the form $(x \pm a)^4 + (x \pm b)^4 = c$

Substitute $y = x \pm \frac{a+b}{2}$ for simple form

Given equation is $(x + 1)^4 + (x + 3)^4 = 16$ ...(i)

Putting $y = x + \frac{1+3}{2}$ *i.e.*, $y = x + 2$

$\therefore\ x + 1 = y - 1$ and $x + 3 = y + 1$

$\therefore$ From (i) we have

$(y - 1)^4 + (y + 1)^4 = 16$

$\Rightarrow\ 2(y^4 + 6y^2 + 1) = 16$

$\Rightarrow\ y^4 + 6y^2 - 7 = 0$

$\Rightarrow\ (y^2 + 7)\ (y^2 - 1) = 0$

$\Rightarrow\ y = \pm 1, y = \pm i\sqrt{7}$

$\Rightarrow\ x + 2 = \pm 1, x + 2 = \pm i\sqrt{7}$

$\Rightarrow\ x = \pm 1 - 2, x = -2 \pm i\sqrt{7}$

$\Rightarrow\ x = -1, -3, x = -2 \pm i\sqrt{7}$

19. (b, d) : Given equation is $|x|^{\frac{4}{5}} - |x|^{\frac{2}{5}} - 12 = 0$

$\Rightarrow\ t^2 - t - 12 = 0$, where $t = |x|^{\frac{2}{5}}$

$\Rightarrow\ t = 4, t = -3$ which is not possible as $|x|^{\frac{2}{5}} \geq 0$

$\Rightarrow\ |x|^{\frac{2}{5}} = 4\ \Rightarrow\ |x|^{\frac{1}{5}} = 2$

$\Rightarrow\ x = \pm (2)^5 = 2^5, -2^5$

$\therefore$ Product of real roots $= (2^5)(-2)^5 = -2^{10} = -1024$

20. (a, b, c, d) : We know that

$(1 + x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + ... + c_nx^n$ ...(*)

Putting $x = 1, -1$ in (*), we have

$c_0 + c_1 + c_2 + c_3 + ... + c_n = 2^n$ ...(i)

and $c_0 - c_1 + c_2 - c_3 + ... + (-1)^n c_n = 0$ ...(ii)

Adding (i) and (ii), we get

$2(c_0 + c_2 + c_4 + ...) = 2^n$

$\Rightarrow\ c_0 + c_2 + c_4 + ... = 2^{n-1}$ ...(iii)

On subtracting (ii) from (i), we get

$2(c_1 + c_3 + c_5 + ...) = 2^n$

$\therefore\ c_1 + c_3 + c_5 + ... = 2^{n-1}$ ...(iv)

Now putting $x = i$ in (*), we get

$(c_0 - c_2 + c_4 + c_6 - ...) + i(c_1 - c_3 + c_5 - c_7 + ...)$

$$= (1 + i)^n = (\sqrt{2})^n \left(\frac{1+i}{\sqrt{2}}\right)^n$$

$\therefore\ (c_0 - c_2 + c_4 - c_6 + ...) + i(c_1 - c_3 + c_5 - c_7 + ...)$

$$= 2^{\frac{n}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

$\Rightarrow\ (c_0 - c_2 + c_4 - c_6 + ...) + i(c_1 - c_3 + c_5 - c_7 + ...)$

$$= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$\Rightarrow \begin{cases} c_0 - c_2 + c_4 - c_6 + ... = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} & \text{...(v)} \\ c_1 - c_3 + c_5 - c_7 + ... = 2^{\frac{n}{2}}\sin\frac{n\pi}{4} & \text{...(vi)} \end{cases}$$

Now, adding (iii) and (v), we get

$$2(c_0 + c_4 + c_8 + ...) = 2^{n-1} + 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

$\Rightarrow\ c_0 + c_4 + c_8 + ... = 2^{n-2} + 2^{\frac{n-2}{2}}\cos\frac{n\pi}{4}$ ...(vii)

Subtracting (v) from (iii), we get

$$2(c_2 + c_6 + c_{10} + ...) = 2^{n-1} - 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

$\Rightarrow\ c_2 + c_6 + c_{10} + ... = 2^{n-2} - 2^{\frac{n-2}{2}}\cos\frac{n\pi}{4}$ ...(viii)

Now, adding (iv) and (vi), we get

$$2(c_1 + c_5 + c_9 + ...) = 2^{n-1} + 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$$

$\Rightarrow\ c_1 + c_5 + c_9 + ... = 2^{n-2} + 2^{\frac{n-2}{2}}\sin\frac{n\pi}{4}$ ...(ix)

Subtracting (vi) from (iv), we get

$$2(c_3 + c_7 + c_{11} + ...) = 2^{n-1} - 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$$

$\Rightarrow\ c_3 + c_7 + c_{11} + ... = 2^{n-2} - 2^{\frac{n-2}{2}}\sin\frac{n\pi}{4}$ ...(x)

Again $(1 + x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + ... + c_nx^n$

Putting $x = 1, \omega, \omega^2$, we get

$\therefore\ 2^n = (c_0 + c_3 + c_6 + ...) + 1(c_2 + c_5 + c_8 + ...) + 1(c_1 + c_4 + c_7 + ...)$

$(1 + \omega)^n = (c_0 + c_3 + c_6 + ...) + \omega^2(c_2 + c_5 + c_8 + ...) + \omega(c_1 + c_4 + c_7 + ...)$

$$(1+\omega^2)^n = (c_0 + c_3 + c_6 + \ldots) + \omega(c_2 + c_5 + c_8 + \ldots) + \omega^2(c_1 + c_4 + c_7 + \ldots)$$

Adding the above relation we get

$$2^n + (1+\omega)^n + (1+\omega^2)^n = 3(c_0 + c_3 + c_6) + (1+\omega+\omega^2)(c_2 + c_5 + c_8 + \ldots) + (1+\omega+\omega^2)(c_1 + c_4 + c_7 + \ldots)$$

$\therefore\quad 3(c_0 + c_3 + c_6 + \ldots) = 2^n + (-\omega^2)^n + (-\omega)^n$

$$= 2^n + (-1)^n\left[\left(\cos\frac{4n\pi}{3} + i\sin\frac{4n\pi}{3}\right) + \left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right)\right]$$

$$= 2^n + (-1)^n\left[\left(\cos\frac{4n\pi}{3} + \cos\frac{2n\pi}{3}\right) + i\left(\sin\frac{4n\pi}{3} + \sin\frac{2n\pi}{3}\right)\right]$$

$$= 2^n + (-1)^n\left[2\cos n\pi\cdot\cos\frac{n\pi}{3} + 2i\left(\sin n\pi\cdot\sin\frac{n\pi}{3}\right)\right]$$

$$= 2^n + 2(-1)^n(-1)^n\cos\frac{n\pi}{3} + 2i\times 0$$

$$\therefore\quad c_0 + c_3 + c_6 + \ldots = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

Also, $c_1 + c_4 + c_7 + \ldots = \frac{1}{3}\left[2^n - \cos\frac{n\pi}{2} + \sqrt{3}\sin\frac{n\pi}{3}\right]$

21. (d) : We have,

$$(7+4\sqrt{3})^{x^2+2x+1} + (7-4\sqrt{3})^{x^2+2x-1} = \frac{14}{7-4\sqrt{3}}$$

$$\Rightarrow\quad (7+4\sqrt{3})(7+4\sqrt{3})^{x^2+2x} + \frac{(7-4\sqrt{3})^{x^2+2x}}{7-4\sqrt{3}} = \frac{14}{7-4\sqrt{3}}$$

$\Rightarrow\quad (7+4\sqrt{3})^{x^2+2x} + (7-4\sqrt{3})^{x^2+2x} = 14$ which is of the form $a^{f(x)} + b^{f(x)} = a + b$ and satisfies $a + b = 14$ and $a\cdot b = 1$

$\therefore\quad f(x) = \pm 1$ where $f(x) = x^2 + 2x$

$\Rightarrow\quad x^2 + 2x = \pm 1$

$\Rightarrow\quad (x+1)^2 = \pm 1 + 1 = 2, 0$

$\Rightarrow\quad (x+1)^2 = 2$ and $(x+1)^2 = 0$

$\therefore\quad x = \pm\sqrt{2} - 1$ and $x = -1$

$\therefore\quad x = \sqrt{2} - 1, -(\sqrt{2}+1)$ and -1

22. (c) : Here $\sqrt{a\sqrt{a\sqrt{a\ldots\infty}}} = a^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty} = a, a > 0$

and $\sqrt{x\sqrt{x\sqrt{x\ldots\infty}}} = x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty} = x$ and $x > 0$

Also $\sqrt{7+4\sqrt{3}} = \sqrt{(2+\sqrt{3})^2} = 2+\sqrt{3}$

Now, $\left(\sqrt{7+4\sqrt{3}}\right)^{\sqrt{a\sqrt{a\sqrt{a\ldots\infty}}}} + (2-\sqrt{3})^{x^2+x-5-\sqrt{x\sqrt{x\sqrt{x\ldots\infty}}}} = 4$

$\Rightarrow\quad (2+\sqrt{3})^a + (2-\sqrt{3})^{x^2-x-5+x} = 4$

$\Rightarrow\quad (2+\sqrt{3})^{x^2-5} + (2-\sqrt{3})^{x^2-5} = 4 = \left[(2+\sqrt{3}) + (2-\sqrt{3})\right]$

which is of the form $A^{f(x)} + B^{f(x)} = A + B$

$\therefore\quad f(x) = \pm 1$

$\Rightarrow\quad x^2 - 5 = \pm 1$

$\Rightarrow\quad x^2 = \pm 1 + 5 = 6, 4$

$\Rightarrow\quad x = \pm\sqrt{6}, x = \pm 2$

$\Rightarrow\quad x = \sqrt{6}$ as $x > \sqrt{5}$ because $x^2 - 5 = a > 0$

23. (c) : Given, α, β are the roots of the equation $x^2 - 8x + 1 = 0$

$$\Rightarrow\quad \alpha, \beta = \frac{8 \pm \sqrt{64-4}}{2} = 4+\sqrt{15}, 4-\sqrt{15}$$

$\therefore\quad \alpha = 4 + \sqrt{15}, \beta = 4 - \sqrt{15}$

Now, $\alpha^{y^2-2y+1} + \beta^{y^2-2y-1} = \frac{50}{7(4-\sqrt{15})}$

$$\Rightarrow\quad (4+\sqrt{15})^{y^2-2y+1} + (4-\sqrt{15})^{y^2-2y-1} = \frac{50}{7(4-\sqrt{15})}$$

$$\Rightarrow\quad (4+\sqrt{15})^{y^2-2y} + (4-\sqrt{15})^{y^2-2y} = \frac{50}{7}$$

Let $(4+\sqrt{15})^{y^2-2y} = t$

$\therefore\quad t + \frac{1}{t} = \frac{50}{7} \Rightarrow 7t^2 - 50t + 7 = 0 \Rightarrow t = 7, 7^{-1}$

$\Rightarrow\quad (4+\sqrt{15})^{y^2-2y} = 7, 7^{-1}$

$\Rightarrow\quad (y^2-2y)\log_{(4+\sqrt{15})}(4+\sqrt{15}) = \log_{4+\sqrt{15}}(7), -\log_{4+\sqrt{15}}(7)$

(Taking log at base $(4+\sqrt{15})$)

$\Rightarrow\quad y^2 - 2y = \log_{4+\sqrt{15}}(7), -\log_{4+\sqrt{15}}(7)$

$\Rightarrow\quad (y-1)^2 = 1 + \log_{4+\sqrt{15}}(7), 1 - \log_{4+\sqrt{15}}(7)$

$\Rightarrow\quad y = 1 \pm \sqrt{1 + \log_{4+\sqrt{15}}(7)}$

(rejecting $1 - \log_{4+\sqrt{15}}(7) < 0$)

Hence, number of solutions are 2.

24. (d) : We have, $(17 + 12\sqrt{2})^t + (17 - 12\sqrt{2})^t = 34$, where $t = x^2 - 2|x|$

Let $a + \sqrt{b} = 17 + 12\sqrt{2}$, $a - \sqrt{b} = 17 - 12\sqrt{2}$ which satisfies $a^2 - b = 1$ $\quad[\because (a+\sqrt{b})(a-\sqrt{b}) = 1$ and $(a+\sqrt{b})^1 = (a-\sqrt{b})^{-1}$ and vice versa$]$

Now, $(17 + 12\sqrt{2})^t + (17 - 12\sqrt{2})^t = 34$

$\therefore\quad t = \pm 1$

$\Rightarrow\quad x^2 - 2|x| = \pm 1$

$\Rightarrow\quad |x|^2 - 2|x| = \pm 1$

$\Rightarrow\quad (|x| - 1)^2 = 1 \pm 1 = 2, 0$

$\Rightarrow\quad |x| - 1 = \pm\sqrt{2}, 0$

$\Rightarrow\quad |x| = (1 \pm \sqrt{2}), 1$

$\Rightarrow\quad |x| = 1 + \sqrt{2}, 1 \qquad (\because |x| \geq 0)$

$\Rightarrow \quad x = \pm(1+\sqrt{2}), \pm 1$

$\therefore$ Number of solutions of the given equation are 4.

25. P $\to$ 2, Q $\to$ 4, R $\to$ 2, S $\to$ 2, T $\to$ 3, U $\to$ 1

(P) Here, $x + \dfrac{1}{x} = \sqrt{3}$

$\Rightarrow \quad x^2 - \sqrt{3}x + 1 = 0$

$\Rightarrow \quad x = \dfrac{\sqrt{3} \pm \sqrt{3-4}}{2}$

$\Rightarrow \quad x = \dfrac{\sqrt{3}}{2} \pm i\left(\dfrac{1}{2}\right)$

$\Rightarrow \quad x = \dfrac{\sqrt{3}}{2} + i\left(\dfrac{1}{2}\right)$ and $x = \dfrac{\sqrt{3}}{2} - i\left(\dfrac{1}{2}\right)$

When $x = \dfrac{\sqrt{3}}{2} + \dfrac{i}{2}$

$\Rightarrow \quad x = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}$

$\Rightarrow \quad x^{2018} = \cos\dfrac{2018\pi}{6} + i\sin\dfrac{2018\pi}{6}$

$\Rightarrow \quad \dfrac{1}{x^{2018}} = \cos\dfrac{2018\pi}{6} - i\sin\dfrac{2018\pi}{6}$

$\therefore \quad x^{2018} + \dfrac{1}{x^{2018}} = 2\cos 2018\dfrac{\pi}{6} = 2\cos\left(336\pi + \dfrac{2\pi}{6}\right)$

$= 2\cos\left(336\pi + \dfrac{\pi}{3}\right) = 2\cos\left(2\pi(168) + \dfrac{\pi}{3}\right)$

$= 2\cos\dfrac{\pi}{3} = 1$

When, $x = \dfrac{\sqrt{3}}{2} - \dfrac{i}{2} = \dfrac{\sqrt{3}}{2} + \left(-\dfrac{1}{2}\right)i$

$\Rightarrow \quad x = \cos\left(-\dfrac{\pi}{6}\right) + i\sin\left(-\dfrac{\pi}{6}\right)$

$\Rightarrow \quad x^{2018} = \cos\dfrac{2018\pi}{6} + i\sin\left(\dfrac{-2018\pi}{6}\right)$

$\Rightarrow \quad \dfrac{1}{x^{2018}} = \cos\dfrac{2018\pi}{6} - i\sin\left(\dfrac{-2018\pi}{6}\right)$

$\therefore \quad x^{2018} + \dfrac{1}{x^{2018}} = 2\cos\dfrac{2018\pi}{6} = 2\cos\dfrac{\pi}{3} = 1$

(Q) $z = 1 + i\sqrt{3} = 2\left(\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right) = -2\omega^2$ and $\bar{z} = -2\omega$

$\therefore \quad (z)^6 + (\bar{z})^6 = 2^6 + 2^6$

$\therefore \quad \dfrac{(z)^6 + (\bar{z})^6}{2^5} = \dfrac{128}{32} = 4$

(R) Since, $1, \alpha_1, \alpha_2, ..., \alpha_{n-1}$ are the n n^{th} roots of unity, it means these are roots of the equation $x^n - 1 = 0$

Now, $x^n - 1 = (x-1)(x-\alpha_1) ... (x - \alpha_{n-1})$

$\Rightarrow \quad \dfrac{x^n - 1}{x - 1} = (x-\alpha_1)(x-\alpha_2) ... (x-\alpha_{n-1})$

$\Rightarrow \quad 1 + x^1 + x^2 + ... + x^{n-1} = (x-\alpha_1)(x-\alpha_2) ... (x-\alpha_{n-1})$

Putting $x = -1$ on both sides, we get

$1 + (-1)^1 + (-1)^2 + ... (-1)^{n-1}$
$= (-1)^{n-1}(1+\alpha_1)(1+\alpha_2) ... (1+\alpha_{n-1})$ ...(*)

$\therefore \quad (1+\alpha_1)(1+\alpha_2) ... (1+\alpha_{n-1}) = 1$

(as in the left of (*) there are n terms and n is odd)

(S) Given, $z - \dfrac{1}{z} = i \Rightarrow z^2 - iz - 1 = 0 \Rightarrow z = \dfrac{i \pm \sqrt{3}}{2}$

$\Rightarrow \quad z = \dfrac{i+\sqrt{3}}{2}$ and $z = \dfrac{i - \sqrt{3}}{2}$

$\therefore \quad z = e^{\frac{i\pi}{6}}$ and $z = e^{i\frac{5\pi}{6}}$

$\therefore \quad z^{2018} = e^{i2018\frac{\pi}{6}}$ and $z^{-2018} = e^{-i2018\frac{\pi}{6}}$

$\therefore \quad z^{2018} + \dfrac{1}{z^{2018}} = 2\cos\dfrac{2018\pi}{6}$

$= 2\cos\left(336\pi + \dfrac{2\pi}{6}\right) = 2\cos\dfrac{\pi}{3} = 1$

Also, $z^{2018} = e^{i\left(\frac{2018\times5\pi}{6}\right)} = e^{i\left(1680\pi + \frac{10\pi}{6}\right)}$

$\therefore \quad z^{2018} = e^{i\frac{10\pi}{6}} = e^{i\frac{5\pi}{3}}$

$\therefore \quad z^{2018} + \dfrac{1}{z^{2018}} = 2\cos\dfrac{5\pi}{3} = 2\left(\dfrac{1}{2}\right) = 1$

(T) Let $\alpha = |z+3| = |(z+4i) + (3-4i)|$
$\geq ||3-4i| - |z+4i||$
$\geq 5 - 2 = 3$

(U) Here, $x = -2 - i\sqrt{3}$

$\Rightarrow \quad x + 2 = -i\sqrt{3}$

$\Rightarrow \quad (x+2)^2 = -3$

$\Rightarrow \quad x^2 + 4x + 7 = 0$

Now, $2x^4 + 5x^3 + 7x^2 - x + 37$
$= 2x^2(x^2+4x+7) - 3x(x^2+4x+7) + 5(x^2+4x+7) + 2$
$= 2 \qquad (\because x^2 + 4x + 7 = 0)$

26. (103) : $\because \ (a\omega^2 - b\omega)(a\omega - b\omega^2) = a^2 + ab + b^2$

$\therefore \quad (a\omega^2 - b\omega)(a\omega - b\omega^2) = \dfrac{(a-b)(a^2+ab+b^2)}{a-b}$

$= \dfrac{a^3 - b^3}{a-b} = \dfrac{4223}{41} = 103$

27. (7) : Let $z = e^{i\theta} = \cos\theta + i\sin\theta$, then

$$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$$

$$z^2 = e^{2i\theta} = \cos 2\theta + i\sin 2\theta, \text{ then}$$

$$\frac{1}{z^2} = e^{-2i\theta} = \cos 2\theta - i\sin 2\theta$$

$$z^9 = e^{9i\theta} = \cos 9\theta + i\sin 9\theta, \text{ then}$$

$$\frac{1}{z^9} = e^{-9i\theta} = \cos 9\theta - i\sin 9\theta$$

Again, $z + \frac{1}{z} = 2\cos\theta$ and $z - \frac{1}{z} = 2i\sin\theta$

Similarly, we can calculate the various results as we needed like

$$z^2 + \frac{1}{z^2}, z^2 - \frac{1}{z^2}, \ldots, z^9 + \frac{1}{z^9}, z^9 - \frac{1}{z^9} \text{ etc.}$$

Now consider, $(2\cos\theta)^4 (2i\sin\theta)^5$

$$= \left(z+\frac{1}{z}\right)^4 \left(z-\frac{1}{z}\right)^5 = \left(z^2-\frac{1}{z^2}\right)^4 \left(z-\frac{1}{z}\right)$$

$\therefore\ 2^9 i\cos^4\theta\sin^5\theta$

$$= \left[\left(z^8+\frac{1}{z^8}\right)-4\left(z^4+\frac{1}{z^4}\right)+6\right]\left(z-\frac{1}{z}\right)$$

$$= z\left(z^8+\frac{1}{z^8}\right)-4z\left(z^4+\frac{1}{z^4}\right)-\frac{1}{z}\left(z^8+\frac{1}{z^8}\right)$$
$$+\frac{4}{z}\left(z^4+\frac{1}{z^4}\right)+6\left(z-\frac{1}{z}\right)$$

$$= \left(z^9-\frac{1}{z^9}\right)-\left(z^7-\frac{1}{z^7}\right)-4\left(z^5-\frac{1}{z^5}\right)$$
$$+4\left(z^3-\frac{1}{z^3}\right)+6\left(z-\frac{1}{z}\right)$$

$\Rightarrow\ 2^8(2i)\cos^4\theta\sin^5\theta = 2i\sin 9\theta - 2i\sin 7\theta - 8i\sin 5\theta + 8i\sin 3\theta + 12i\sin\theta$

$= 2i(\sin 9\theta - \sin 7\theta - 4\sin 5\theta + 4\sin 3\theta + 6\sin\theta)$ ...(i)

Also, $2^8\cos^4\theta\sin^5\theta = a\sin 9\theta + b\sin 7\theta - c\sin 5\theta + d\sin 3\theta + e\sin\theta$ ...(ii) [Given]

From (i) & (ii), we have

$a = 1, b = -1, c = 4, d = 4, e = 6$

$\therefore\ a^2 + b^2 + c^2 + d^2 + e^2 = 1 + 1 + 16 + 16 + 36 = 70$

Given, $a^2 + b^2 + c^2 + d^2 + e^2 = 10\lambda$. Thus, $\lambda = 7$

28. (1728) : Let $z = a + b\omega + c\omega^2$

$\Rightarrow\ \bar{z} = \overline{a + b\omega + c\omega^2} = a + b\omega^2 + c\omega$

Now, $|z|^2 = z\bar{z} = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= a^2 + b^2 + c^2 + ab(\omega + \omega^2) + bc(\omega + \omega^2) + ca(\omega + \omega^2)$

$= a^2 + b^2 + c^2 - ab - bc - ca$

$= \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$

$\therefore\ |z| = \frac{1}{\sqrt{2}}\sqrt{(a-b)^2+(b-c)^2+(c-a)^2}$

Again $|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega| = |z| + |\bar{z}| = 2|z|$

$\therefore\ |a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$

$= \sqrt{2}\sqrt{(a-b)^2+(b-c)^2+(c-a)^2} \geq \sqrt{2}\times\sqrt{(-1)^2+(-1)^2+2^2}$

($\because\ b = a + 1, c = b + 1$ which means a, b, c are consecutive)

$= (12)^{1/2} = (12^3)^{1/6} = (1728)^{1/6} = \alpha^{1/6}$ (Given)

$\therefore\ \alpha = 1728$

29. (1) : Given, $2^x + 3^x + 6^x - 4^x - 9^x = 1$

$\Rightarrow\ 2^x + 3^x + 6^x - 4^x - 9^x - 1 = 0$ ...(i)

$\Rightarrow\ 2^x + 3^x + 2^x \cdot 3^x - 2^{2x} - 3^{2x} - 1 = 0$

$\Rightarrow\ a + b + ab - a^2 - b^2 - 1 = 0$, where $a = 2^x, b = 3^x$

$\Rightarrow\ a^2 + b^2 - ab - a - b + 1 = 0$

$\Rightarrow\ 2(a^2 + b^2 - ab - a - b + 1) = 0$

$\Rightarrow\ (a^2 - 2a + 1) + (a^2 + b^2 - 2ab) + (b^2 - 2b + 1) = 0$

$\Rightarrow\ (a - 1)^2 + (a - b)^2 + (b - 1)^2 = 0$

$\Rightarrow\ a - 1 = 0, a - b = 0, b - 1 = 0$

(If sum of the square of two or more numbers is zero, then they are independently zero)

$\Rightarrow\ a = b = 1$

$\therefore\ 2^x = 1, 3^x = 1, 2^x = 3^x$

$\Rightarrow\ 2^x = 2^0, 3^x = 3^0$ [$\because\ 2^x = 3^x$ is not possible]

$\Rightarrow\ x = 0$ is the only solution of (i)

$\therefore$ Number of solution is only 1.

30. (0) : Given, $z^3 - (1 + i)z^2 + (1 + i)z - i = 0$

$\Rightarrow\ z^2(z - i) - z(z - i) + (z - i) = 0$

$\Rightarrow\ (z - i)(z^2 - z + 1) = 0$

$\Rightarrow\ (z - i)(z + \omega)(z + \omega^2) = 0$

$\Rightarrow\ z = i, -\omega, -\omega^2$

Now, $z^{2017} + z^{2018} - 1$ ...(i)

$= i^{2017} + i^{2018} - 1$

$= i\, i^{4\lambda_1} + i^2\, i^{4\lambda_1} - 1$ (Here, $\lambda_1 = 504$)

$= i - 1 - 1 \neq 0$

$\therefore\ i$ is not a root of the equation.

Again putting $z = -\omega$ in (i), we have

$(-\omega)^{2017} + (-\omega)^{2018} - 1$

$= -\omega \cdot \omega^{3(672)} + \omega^2 \cdot \omega^{3(672)} - 1$

$= -\omega + \omega^2 - 1 = 2\omega^2 \neq 0$, so $z = -\omega$ is also not a root of the equation.

Now, putting $z = -\omega^2$, we have

$(-\omega^2)^{2017} + (-\omega^2)^{2018} - 1$

$= -\omega^{4034} + \omega^{4036} - 1$

$= -\omega^2 + \omega - 1$

$= 2\omega \neq 0$, so $z = -\omega^2$ is not a root of the equation.

$\therefore$ Number of common solutions of the equation is 0.

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MOCK TEST PAPER 2019
JEE MAIN

Time : 1 hr 15 min. **Series - 5**

The entire syllabus of Mathematics of JEE MAIN is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below:

	Topic	Syllabus In Details
UNIT NO. 5	Differential calculus	Functions, Limits, continuity & differentiability.
	Statistics & Probability	Measures of Dispersion: Calculation of mean, median, mode of grouped and ungrouped data, calculation of standard deviation, variance and mean deviation for grouped and ungrouped data. Probability : Probability of an event, addition and multiplication theorems of probability.
	Co-ordinate geometry-3D	Coordinate axes and coordinate planes in three dimensions. Coordinate of a point. Distance between two points and section formula

1. Let f be a real valued function satisfying $f(x+y) = f(x)\,f(y)$ for all $x, y \in R$ such that $f(1) = 2$. If $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, then a =

(a) 3 (b) 4
(c) 2 (d) none of these

2. If $\{x\}$ and $[x]$ denote respectively the fractional and integral parts of a real number x, then the number of solutions of the equation $4\{x\} = x + [x]$, is

(a) 1 (b) 2
(c) 3 (d) infinitely many

3. The domain of the definition of the function $f(x) = \log_4[\log_5\{\log_3(18 - x^2 - 77)\}]$, is

(a) (8, 10) (b) [8, 10] (c) $(-\infty, 8]$ (d) $[10, \infty)$

4. The period of $f(x) = \cos(\cos x) + \cos(\sin x)$ is

(a) π (b) 2π (c) $\pi/2$ (d) 4π

5. A function $f(x) = \begin{cases} x^2 \sin\dfrac{\pi x}{2}, & |x| < 1 \\ x|x|, & |x| \ge 1 \end{cases}$ is

(a) an even function (b) an odd function
(c) a periodic function (d) none of these

6. If $f(x) = x\dfrac{e^{[x]+|x|} - 2}{[x]+|x|}$, then $\lim\limits_{x\to 0} f(x)$, is

(a) -1 (b) 0
(c) 1 (d) non-existent

7. The value of $\lim\limits_{x\to -\infty}\left\{\dfrac{x^4 \sin\left(\dfrac{1}{x}\right) + x^2}{1 + |x^3|}\right\}$ is

(a) 1 (b) -1 (c) 0 (d) ∞

8. If $f(x) = \begin{cases} \dfrac{\sin\{\cos x\}}{x - \dfrac{\pi}{2}}, & x \ne \dfrac{\pi}{2} \\ 1, & x = \dfrac{\pi}{2} \end{cases}$

where $\{\cdot\}$ represents the fractional part function, then $f(x)$ is

(a) continuous at $x = \pi/2$
(b) $\lim\limits_{x\to \pi/2} f(x)$ exists, but $f(x)$ is not continuous at $x = \pi/2$
(c) $\lim\limits_{x\to \pi/2} f(x)$ does not exists
(d) $\lim\limits_{x\to \pi/2^-} f(x) = 1$

By : Sankar Ghosh, S.G.M.C, Mob. 09831244397.

9. Let $f(x) = [|x|]$, where $[\cdot]$ denotes the greatest integer function, then $f'(-1)$ is
(a) 0 (b) 1
(c) non-existent (d) none of these

10. Let $f(x) = \min\{1, \cos x, 1 - \sin x\}$, $-\pi \le x \le \pi$. Then, $f(x)$ is
(a) not continuous at $x = \pi/2$
(b) continuous but not differentiable at $x = 0$
(c) neither continuous nor differentiable at $x = \pi/2$
(d) none of these

11. Let f be a differentiable function satisfying the condition:
$$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)} \text{ for all } x, y \in R (y \ne 0) \text{ and } f(y) \ne 0$$
If $f'(1) = 2$, then $f'(x)$ is equal to
(a) $2f(x)$ (b) $\frac{f(x)}{x}$ (c) $2x\,f(x)$ (d) $\frac{2f(x)}{x}$

12. Let a function $f(x)$ defined on [3, 6] be given by
$$f(x)=\begin{cases}\log_e[x] & , 3 \le x < 5\\ |\log_e x| & , 5 \le x < 6\end{cases}$$
Then $f(x)$ is
(a) continuous and differentiable on [3, 6]
(b) continuous on [3, 6) but not differentiable at $x = 4, 5$
(c) differentiable on [3, 6) but not continuous at $x = 4, 5$
(d) none of these

13. Let $f(x) = \sin x$, $g(x) = [x + 1]$ and $h(x) = gof(x)$, where $[\cdot]$ is the greatest integer function. Then, $h'\left(\frac{\pi}{2}\right)$ is
(a) 1 (b) -1
(c) non-existent (d) none of these

14. Let $f(x)$ be a polynomial satisfying
$(f(\alpha))^2 + (f'(\alpha))^2 = 0$. Then, $\lim_{x\to\alpha} \frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right] =$
(where $[\cdot]$ denotes the greatest integer function)
(a) 0 (b) 1 (c) -1 (d) $f(\alpha)f'(\alpha)$

15. If $f(x)=\frac{9^x}{9^x+9}$,
then $f\left(\frac{1}{2019}\right)+f\left(\frac{2}{2019}\right)+f\left(\frac{3}{2019}\right)+\ldots+\left(\frac{4037}{2019}\right)=$
(a) 1009 (b) $\frac{4037}{2}$ (c) 2018 (d) 2019

16. The graph of the function $y = f(x)$ has a unique tangent at the point $(a, 0)$ through which the graph passes. Then $\lim_{x\to a} \frac{\log_e(1+6f(x))}{3f(x)}$ is
(a) 1 (b) 2
(c) 0 (d) none of these

17. Two integers x and y are chosen with replacement out of the set {0, 1, 2, 3, ..., 10}. Then the probability that $|x - y| > 5$ is
(a) $\frac{81}{121}$ (b) $\frac{30}{121}$ (c) $\frac{25}{121}$ (d) $\frac{20}{121}$

18. A is one of 6 horses entered for a race, and is to be ridden by one of two jockeys B and C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win. If C rides A, his chances of winning is tripled. What are the odds against winning of A ?
(a) 5 : 13 (b) 5 : 18
(c) 13 : 5 (d) none of these

19. The probability that $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$ is an integer $x, y \in \{1, 2, 3, 4\}$, is
(a) $\frac{1}{16}$ (b) $\frac{3}{16}$
(c) $\frac{15}{16}$ (d) none of these.

20. Three numbers are chosen at random without replacement from 1, 2, 3, …, 10. The probability that the minimum of the chosen numbers is 4 or their maximum is 8 is
(a) $\frac{11}{40}$ (b) $\frac{3}{10}$
(c) $\frac{1}{40}$ (d) none of these.

21. Let the probability P_n that a family has exactly n children be αp^n when $n \ge 1$ and $P_0 = 1 - \alpha p(1 + p + p^2 + \ldots)$. Suppose that all sex distributions of n children have the same probability. If $k \ge 1$, then the probability that a family contains exactly k boys is
(a) $\frac{2\alpha}{(2-p)^{k+1}}$ (b) $\frac{p^k}{(2-p)^{k+1}}$
(c) $\frac{2\alpha \cdot p}{(2-p)^{k+1}}$ (d) none of these

22. The mean and median of 100 items are 50 and 52 respectively. The value of largest item is 100. It was later found that it is 110 and not 100. The true mean and median are
(a) 50.10, 51.5 (b) 50.10, 52
(c) 50, 51.5 (d) none of these

23. If the standard deviation of the observations $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ is $\sqrt{10}$. The standard deviation of observations 15, 16, 17,18, 19, 20, 21, 22, 23, 24, 25 will be

(a) $\sqrt{10}+20$ (b) $\sqrt{10}+10$
(c) $\sqrt{10}$ (d) none of these

24. An aeroplane flies around a square, the sides of which measure 100 miles each. The aeroplane covers at a speed of 100 mph the first side, at 200 mph the second side, at 300 mph the third side and 400 mph the fourth side. The average speed of the aeroplane around the square is
(a) 190 mph (b) 195 mph
(c) 192 mph (d) 200 mph.

25. The first of the two samples has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and standard deviation $\sqrt{13.44}$, then the standard deviation of the second group is
(a) 4 (b) 5 (c) 6 (d) 7

26. From a sample of n observations, the arithmetic mean and variance are calculated. It is then found that one of the values, x_1 is in error and should be replaced by x_1'. The adjustment to the variance to correct this error is

(a) $\frac{1}{n}(x_1'-x_1)\left(x_1'+x_1-\frac{x_1'-x_1+2T}{n}\right)$

(b) $\frac{1}{n}(x_1'-x_1)\left(x_1'+x_1+\frac{x_1'-x_1+2T}{n}\right)$

(c) $\frac{1}{n}(x_1'+x_1)\left(x_1'-x_1+\frac{x_1'-x_1+2T}{n}\right)$

(d) none of these

27. The xy-plane divides the line joining the points (–1, 3, 4) and (2, –5, 6)
(a) internally in the ratio 2 : 3
(b) externally in the ratio 2 : 3
(c) internally in the ratio 3 : 2
(d) externally in the ratio 3 : 2

28. The points $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$ are the vertices of a
(a) trapezium (b) rectangle
(c) rhombus (d) square

29. In a ΔABC, the mid-point of the sides AB, BC and CA are respectively $(l, 0, 0)$, $(0, m, 0)$ and $(0, 0, n)$. Then $\frac{AB^2+BC^2+CA^2}{l^2+m^2+n^2}=$
(a) 2 (b) 4 (c) 8 (d) 16

30. The cosine of the angle of the triangle with vertices $A(1, -1, 2)$, $B(6, 11, 2)$ and $C(1, 2, 6)$ is
(a) $\frac{63}{65}$ (b) $\frac{36}{65}$ (c) $\frac{16}{65}$ (d) $\frac{13}{64}$

SOLUTIONS

1. **(a) :** We have, $f(x)=[f(1)]^x=2^x$ for all $x \in R$.
[If $f: R \to R$ is a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in R$, then $f(x) = \{f(1)\}^x\ \forall\ x \in R$]

$$\therefore \sum_{k=1}^{n} f(a+k)=16(2^n-1)$$

$$\Rightarrow \sum_{k=1}^{n} 2^{a+k}=16(2^n-1) \Rightarrow 2^a\sum_{k=1}^{n} 2^k=16(2^n-1)$$

$$\Rightarrow 2^a(2+2^2+2^3+...+2^n)=16(2^n-1)$$

$$\Rightarrow 2^a\times 2\left(\frac{2^n-1}{2-1}\right)=16(2^n-1)$$

$$\Rightarrow 2^{a+1}=2^4 \Rightarrow a=3$$

2. **(b) :** We know that, $x = [x] + \{x\}$
$\therefore$ $4\{x\} = x + [x] \Rightarrow 4\{x\} = [x] + \{x\} + [x]$
$\Rightarrow 3\{x\} = 2[x]$ ...(1)
But, $0 \le \{x\} < 1$
$\Rightarrow 0 \le 3\{x\} < 3 \Rightarrow 0 \le 2[x] < 3$

$$\Rightarrow 0 \le [x] < \frac{3}{2} \Rightarrow [x]=0,1$$

When, $[x] = 0$, then $\{x\} = 0$ [from (1)]
When $[x] = 1$, then $\{x\} = 2/3$

$$\therefore x=0, \frac{5}{3} \quad [\because x=[x]+\{x\}]$$

3. **(a) :** Here $f(x) = \log_4[\log_5\{\log_3(18x - x^2 - 77)\}]$ will be defined if $[\log_5\{\log_3(18x - x^2 - 77)\}] > 0$
$\Rightarrow \log_3(18x - x^2 - 77) > 5^0$
$\Rightarrow (18x - x^2 - 77) > 3^1 \Rightarrow x^2 - 18x + 80 < 0$
$\Rightarrow (x - 8)(x - 10) < 0 \Rightarrow 8 < x < 10$
$\Rightarrow x \in (8, 10)$

4. **(c) :** Here we have, $f(x) = \cos(\cos x) + \cos(\sin x)$
$\therefore$ $\cos\{\cos(\pi + x)\} = \cos(-\cos x) = \cos(\cos x)$
and $\cos\{\sin(\pi + x)\} = \cos(-\sin x) = \cos(\sin x)$
Therefore, $\cos(\cos x)$ and $\cos(\sin x)$ are periodic functions with period π. So, $f(x)$ should have period π.
But, we find that

$$\cos\left\{\sin\left(\frac{\pi}{2}+x\right)\right\}=\cos(\cos x)$$

and $\cos\left\{\cos\left(\frac{\pi}{2}+x\right)\right\}=\cos(-\sin x)=\cos(\sin x)$

Therefore, $f(x)$ is periodic with period $\pi/2$.

5. (b) : We have, $f(x)=\begin{cases} -x^2, & \text{if } x\le -1 \\ x^2\sin\frac{\pi x}{2}, & \text{if } -1<x<1 \\ x^2, & \text{if } x\ge 1 \end{cases}$

For $x \in (-1, 1)$, we have

$$f(-x)=(-x)^2\sin\left(\frac{-\pi x}{2}\right)=-x^2\sin\frac{\pi x}{2}=-f(x)$$

Let $x \in [1, \infty)$. Then, $x = 1 + k$, $k > 0$

Clearly, $-x = -1 - k \in (-\infty, -1]$

Now, $f(-x) = f(-1 - k) = -(-1 - k)^2 = -(1 + k)^2$

and $f(x) = (1 + k)^2$

$\therefore\ f(-x) = -f(x)\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

Thus, $f(-x) = -f(x)\ \forall\ x \in R$

Hence, $f(x)$ is an odd function.

6. (d) : We have,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x\frac{e^{-1-x}-2}{-1-x}$$

$[\because\ [x] = -1 \text{ and } |x| = -x \text{ when } -1 < x < 0]$

$$\Rightarrow \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} 0\times\frac{e^{-1}-2}{-1}=0$$

and, $\displaystyle\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x\left(\frac{e^x-2}{x}\right)$

$[\because\ [x] = 0 \text{ and } |x| = x \text{ when } 0 \le x < 1]$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} e^x-2=1-2=-1$$

$\therefore\ \displaystyle\lim_{x\to 0} f(x)$ does not exist.

7. (b) : We have, $\displaystyle\lim_{x\to -\infty}\left\{\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{1+|x^3|}\right\}$

$$=\lim_{h\to\infty}\frac{h^4\sin\left(-\frac{1}{h}\right)+h^2}{1+|-h^3|},\quad \text{where } h=-x$$

$$=\lim_{h\to\infty}\frac{-h^4\sin\left(\frac{1}{h}\right)+h^2}{1+h^3}$$

$$=\lim_{h\to\infty}\frac{-h\sin\left(\frac{1}{h}\right)+\frac{1}{h}}{\frac{1}{h^3}+1}=\frac{-1+0}{0+1}=-1$$

8. (b) : We have,

$$\lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)=\lim_{h\to 0}\frac{\sin\left\{\cos\left(\frac{\pi}{2}-h\right)\right\}}{-h}$$

$$=\lim_{h\to 0}\frac{\sin\{\sin h\}}{-h}=-\lim_{h\to 0}\frac{\sin(\sin h)}{\sin h}\times\frac{\sin h}{h}=-1$$

and $$\lim_{x\to\frac{\pi}{2}^+} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)=\lim_{h\to 0}\frac{\sin\left\{\cos\left(\frac{\pi}{2}+h\right)\right\}}{\frac{\pi}{2}+h-\frac{\pi}{2}}$$

$$=\lim_{h\to 0}\frac{\sin\{-\sin h\}}{h}=-\lim_{h\to 0}\frac{\sin(\sin h)}{\sin h}\times\frac{\sin h}{h}=-1$$

So, $\displaystyle\lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{x\to\frac{\pi}{2}^+} f(x)\neq f\left(\frac{\pi}{2}\right)$

Therefore, $\displaystyle\lim_{x\to\pi/2} f(x)$ exists, $f(x)$ is not continuous at $x = \pi/2$.

9. (c) : We have $f(x)=[|x|]=\begin{cases} 0, & \text{when } -1<x<1 \\ 1, & \text{when } -2<x\le -1 \end{cases}$

$$\therefore\ Lf'(-1)=\lim_{h\to 0^-}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0^-}\frac{1-1}{h}=0$$

and $$Rf'(-1)=\lim_{h\to 0^+}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0^+}\frac{0-1}{h}=-\infty$$

Since $Lf'(-1) \neq Rf'(-1)$, therefore $f(x)$ is not differentiable at $x = -1$.

10. (b) : Given that $f(x) = \min\{1, \cos x, 1 - \sin x\}$, $-\pi \le x \le \pi$

$$\Rightarrow\ f(x)=\begin{cases} \cos x, & -\frac{\pi}{2}\le x\le 0 \\ 1-\sin x, & 0<x\le\frac{\pi}{2} \\ \cos x, & \frac{\pi}{2}<x\le\pi \end{cases}$$

Now $\displaystyle\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\cos x=1$

and $\displaystyle\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1-\sin x)=1$

and, $f(0) = \cos 0 = 1$

Clearly $\displaystyle\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$, so $f(x)$ is continuous at $x = 0$.

Again $\displaystyle Lf'(0)=\frac{d}{dx}(\cos x)\Big|_{x=0}=0$

and $Rf'(0)=\frac{d}{dx}(1-\sin x)\Big|_{x=0}=-1$

$\therefore\ Lf'(0) \neq Rf'(0)$

Hence, $f(x)$ is not differentiable at $x = 0$.

Thus $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.

11. (d) : We have,

$$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)} \text{ for all } x, y \in R\ (y \neq 0) \text{ and } f(y) \neq 0$$

$$f(1)=\frac{f(1)}{f(1)} \Rightarrow f(1)=1. [\text{Replacing } x \text{ and } y \text{ both by } 1]$$

Now, $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\Rightarrow f'(x)=f(x)\lim_{h\to 0}\left\{\frac{\frac{f(x+h)}{f(x)}-1}{h}\right\}$$

$$\Rightarrow f'(x)=f(x)\lim_{h\to 0}\left\{\frac{f\left(\frac{x+h}{x}\right)-1}{h}\right\}\left[\because f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}\right]$$

$$\Rightarrow f'(x)=f(x)\lim_{h\to 0}\left\{\frac{f\left(1+\frac{h}{x}\right)-f(1)}{h}\right\}[\because f(1)=1]$$

$$\Rightarrow f'(x)=\frac{f(x)}{x}\lim_{h\to 0}\left\{\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}}\right\}\quad[\because f(1)=1]$$

$$\Rightarrow f'(x)=\frac{f(x)}{x}f'(1)=\frac{2f(x)}{x}\qquad[\because f'(1)=2]$$

12. (d) : We have, $f(x)=\begin{cases}\log_e 3, & 3\le x<4\\ \log_e 4, & 4\le x<5\\ \log_e x, & 5\le x<6\end{cases}$

Clearly, $f(x)$ continuous and differentiable on $[3, 4) \cup (4, 5) \cup (5, 6)$

At $x = 4$, we have

$$\lim_{x\to 4^-} f(x)=\log_e 3 \text{ and } \lim_{x\to 4^+} f(x)=\log_e 4$$

$$\because \lim_{x\to 4^-} f(x)\neq \lim_{x\to 4^+} f(x)$$

Thus, $f(x)$ is neither continuous nor differentiable at $x = 4$.

At $x = 5$, we have

$$\lim_{x\to 5^-} f(x)=\log_e 4 \text{ and } \lim_{x\to 5^+} f(x)=\log_e 5$$

$$\therefore \lim_{x\to 5^-} f(x)\neq \lim_{x\to 5^+} f(x)$$

So, $f(x)$ is neither continuous nor differentiable at $x = 5$.

13. (c) : We are given that

$f(x) = \sin x$, $g(x) = [x + 1]$

and $h(x) = gof(x) = g(\sin x) = [\sin x + 1]$

$$\text{Now, } Lh'\left(\frac{\pi}{2}\right)=\lim_{\delta\to 0^-}\frac{h\left(\frac{\pi}{2}+\delta\right)-h\left(\frac{\pi}{2}\right)}{\delta}$$

$$=\lim_{\delta\to 0^-}\frac{\left[\sin\left(\frac{\pi}{2}+\delta\right)+1\right]-\left[\sin\frac{\pi}{2}+1\right]}{\delta}$$

$$=\lim_{\delta\to 0^-}\frac{1-2}{\delta}\qquad\left[\because \delta\to 0^- \therefore \frac{\pi}{2}+\delta<\frac{\pi}{2}\right]$$

$= \infty$

Similarly, we have $Rh'\left(\frac{\pi}{2}\right)=-\infty$.

Hence, $h'\left(\frac{\pi}{2}\right)$ does not exist.

14. (b) : We are given that the polynomial $f(x)$ satisfies the relation $(f(\alpha))^2 + (f'(\alpha))^2 = 0$

$\therefore\ f(\alpha) = 0 = f'(\alpha)$

$\Rightarrow\ x = \alpha$ is a root of $f(x)$ and $f'(x)$

$\Rightarrow\ (x - \alpha)^2$ is a factor of $f(x)$

Let $f(x) = (x - \alpha)^2\,\phi(x)$.

Then $f'(x)=2(x-\alpha)\phi(x)+(x-\alpha)^2\phi'(x)$.

$$\therefore \frac{f(x)}{f'(x)}=\frac{(x-\alpha)\phi(x)}{2\phi(x)+(x-\alpha)\phi'(x)}$$

$$\text{Now, }\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right]$$

$$=\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)}-\left\{\frac{f'(x)}{f(x)}\right\}\right), \text{ [since } [x] = x - \{x\}]$$

$$=\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\times\frac{f'(x)}{f(x)}-\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left\{\frac{f'(x)}{f(x)}\right\}$$

$= 1 - 0 = 1$

15. (b) : We have $f(x)=\frac{9^x}{9^x+9}$

$$\therefore\ f(2-x)=\frac{9^{2-x}}{9^{2-x}+9}$$

$$\text{So, } f(x)+f(2-x)=\frac{9^x}{9^x+9}+\frac{9^{2-x}}{9^{2-x}+9}=\frac{9^x}{9^x+9}+\frac{9}{9^x+9}=1$$

$\Rightarrow\ f(x) + f(2 - x) = 1$

$$\therefore\ f\left(\frac{1}{2019}\right)+f\left(\frac{2}{2019}\right)+f\left(\frac{3}{2019}\right)+\ldots+f\left(\frac{4037}{2019}\right)$$

$$=\left\{f\left(\frac{1}{2019}\right)+f\left(\frac{4037}{2019}\right)\right\}+\left\{f\left(\frac{2}{2019}\right)+f\left(\frac{4036}{2019}\right)\right\}+\ldots$$

$$+\left\{f\left(\frac{2018}{2019}\right)+f\left(\frac{2020}{2019}\right)\right\}+f\left(\frac{2019}{2019}\right)$$

$$=\{1+1+\ldots+1(2018\text{ times})\}+f(1)=2018+\frac{1}{2}=\frac{4037}{2}$$

16. (b) : According to the problem we have, $f(a) = 0$ and $f(x)$ is differentiable at $x = a$

$$\therefore\ \lim_{x\to a}\frac{\log_e(1+6f(x))}{3f(x)}=\lim_{x\to a}\frac{\frac{6f'(x)}{1+6f(x)}}{3f'(x)}\ \text{[By LH rule]}$$

$$=\frac{2}{1+6f(a)}\ \text{[Since } f(x) \text{ is passes through } (a, 0)]$$

$= 2$ [As $f(a) = 0$]

17. (b) : The total number of selections of two numbers x and y from the numbers 0 to 10 = 11 × 11 = 121

Now, $|x - y| > 5$

The pairs of value (x, y) are

(0, 6), (0, 7), (0, 8), (0, 9), (0 10), (1, 7), (1, 8), (1, 9), (1, 10), (2, 8), (2, 9), (2, 10), (3, 9), (3, 10), (4, 10), (6, 0), (7, 0), (8, 0), (9, 0), (10, 0), (7, 1), (8, 1), (8, 2), (9, 1), (9, 2), (9, 3), (10, 1), (10, 2), (10, 3), (10, 4)

So, there are 30 pairs of values of x and y.

Hence, the required probability $=\frac{30}{121}$.

18. (c) : Let us define the events as

E_1 : Jockey B rides horse A

E_2 : Jockey C rides horse A

E : The horse A wins

$\therefore\ P(E_1)=\frac{2}{3}$ [Since odds in favour of E_1 are 2 : 1]

and $P(E|E_1)=\frac{1}{6}$

Again $P(E_2)=1-P(E_1)=1-\frac{2}{3}=\frac{1}{3}$

and $P(E|E_2)=3P(E|E_1)=\frac{1}{2}$

Now, the required probability = $P(E)$

$= P(E_1 \cap E) + P(E_2 \cap E)$

$= P(E_1)\ P(E|E_1) + P(E_2)\ P(E|E_2)$

$=\frac{2}{3}\cdot\frac{1}{6}+\frac{1}{3}\cdot\frac{1}{2}=\frac{5}{18}$

Therefore the odds against winning of A are 13 : 5.

19. (b) : Clearly x should lie in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and y in $[0, \pi]$ in order to get the integer value of $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$.

$\Rightarrow\ x = 1$ and $y = 1, 2, 3$

$\therefore$ Required probability $=\frac{3}{16}$.

20. (a) : Let us define the events in the following way :

A : 4 being the minimum number

B : 8 being the maximum number

$A \cap B$: 4 being the minimum number and 8 being the maximum number

Therefore $P(A)=\frac{^6C_2}{^{10}C_3}=\frac{15}{120}$

$P(B)=\frac{^7C_2}{^{10}C_3}=\frac{21}{120}$ and $P(A\cap B)=\frac{^3C_2}{^{10}C_3}=\frac{3}{120}$

$\therefore$ The required probability,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$=\frac{15}{120}+\frac{21}{120}-\frac{3}{120}=\frac{33}{120}=\frac{11}{40}$

21. (d) : We are given that $P_n=\alpha p^n,\ n\geq 1$

and $P_0=1-\alpha p(1+p+p^2+......)$.

Now let us define the events in the following way :

E_j = There are j children in the family, $j = 0, 1, 2, ..., n$

A = there are exactly k boys in the family

We have, $P(E_j)=P_j=\alpha p^j;\ j=0,1,2,.....n$

and $P(A|E_j)=\frac{^jC_k}{2^j},\ j\geq k$

Now, $A=\bigcup_{j=1}^{\infty}(A\cap E_j)\ \Rightarrow\ P(A)=P\left(\bigcup_{j=1}^{\infty}(A\cap E_j)\right)$

$$\therefore\ P(A)=\sum_{j=k}^{\infty}P(A\cap E_j)=\sum_{j=k}^{\infty}P(E_j)P(A|E_j)$$

$$=\sum_{j=k}^{\infty}\alpha p^j\left(\frac{^jC_k}{2^j}\right)=\alpha\sum_{j=k}^{\infty}\left(\frac{p}{2}\right)^j\cdot{}^jC_K$$

$$=\alpha\sum_{r=0}^{\infty}{}^{k+r}C_r\left(\frac{p}{2}\right)^{k+r}=\alpha\left(\frac{p}{2}\right)^k\sum_{r=0}^{\infty}{}^{k+r}C_r\left(\frac{p}{2}\right)^r$$

22. (b) : Here $n = 100$, mean = 50, median = 52

$$\therefore\ \bar{x}=\frac{1}{n}\sum_{i=1}^{100}x_i=50\ \Rightarrow\ \sum_{i=1}^{100}x_i=5000$$

Now corrected $\sum_{i=1}^{100}x_i=5000-100+110=5010$

$\therefore$ Corrected mean $=\frac{1}{100}\sum_{i=1}^{100}x_i=\frac{5010}{100}=50.10$

As median is positional average therefore it will remain same.

23. (c) : From the given information, we may write a relation $y = x + 20$, between the two sets of data. [where x denotes the old values and y denotes the new values]

So, standard deviation of $x = \sqrt{10}$

Let $y_i = x_i + 20$ where $i = 1, 2, ..., 11$

$\therefore \quad \bar{y} = \bar{x} + 20$

$$\Rightarrow \quad \frac{1}{n}\sum_{i=1}^{11}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{11}(x_i - \bar{x})^2$$

$$\Rightarrow \quad \sqrt{\frac{1}{n}\sum_{i=1}^{11}(y_i - \bar{y})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{11}(x_i - \bar{x})^2} = \sqrt{10}$$

Thus the standard deviation of y is $\sqrt{10}$.

24. (c) : Total distance covered $= 4 \times 100 = 400$ miles

Total time taken $= \dfrac{100}{100} + \dfrac{100}{200} + \dfrac{100}{300} + \dfrac{100}{400} = \dfrac{25}{12}$ hours

$$\therefore \quad \text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}}$$

$$= \frac{400}{25} \times 12 = 192 \text{ mph}$$

25. (a) : We have $n_1 = 100, \bar{x}_1 = 15, \sigma_1 = 3,$
$n = n_1 + n_2 = 250, \bar{x} = 15.6$ and $\sigma = \sqrt{13.44}$

We have to find σ_2.

Now, $n_2 = 250 - 100 = 150$

We know that, $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$$\Rightarrow \quad 15.6 = \frac{100 \times 15 + 150 \times \bar{x}_2}{250} \Rightarrow \bar{x}_2 = 16$$

Hence $d_1 = \bar{x}_1 - \bar{x} = 15 - 15.6 = -0.6$

and $d_2 = \bar{x}_2 - \bar{x} = 16 - 15.6 = 0.4$

The variance σ^2 of the combined group is given by the formula

$$(n_1 + n_2)\sigma^2 = n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)$$

$$\Rightarrow \quad 250 \times 13.44 = 100(9 + 0.36) + 150(\sigma_2^2 + 0.16)$$

$$\Rightarrow \quad 150\sigma_2^2 = 250 \times 13.44 - 100 \times 9.36 - 150 \times 0.16 = 2400$$

$$\Rightarrow \quad \sigma_2^2 = \frac{2400}{150} = 16 \Rightarrow \sigma_2 = \sqrt{16} = 4$$

26. (a) : Let $x_1, x_2, ..., x_n$ be the given sample of n observations. Their variance σ^2 is given by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n}(x_1^2 + x_2^2 + ... + x_n^2) - \frac{T^2}{n^2},$$

where $T = x_1 + x_2 + ... + x_n$

Let ${\sigma_1}^2$ be the corrected variance, if the wrong observation x_1, is replaced by the corrected value x'.

Then

$$\sigma_1^2 = \frac{1}{n}(x_1'^2 + x_2^2 + ... + x_n^2) - \left\{\frac{T - x_1 + x_1'}{n}\right\}^2$$

Adjustment to the variance to correct error is

$$\sigma_1^2 - \sigma^2 = \frac{1}{n}\{x_1'^2 - x_1^2\} - \frac{1}{n^2}\{(T - x_1 + x_1')^2 - T^2\}$$

$$= \frac{1}{n}(x_1' + x_1)(x_1' - x_1) - \frac{1}{n^2}\{(x_1' - x_1) \times (2T - x_1 + x_1')\}$$

$$= \frac{1}{n}(x_1' - x_1)\left(x_1' + x_1 - \frac{x_1' - x_1 + 2T}{n}\right)$$

27. (b) : The xy-plane divides the line segment joining the points $(-1, 3, 4)$ and $(2, -5, 6)$ in the ratio $-4 : 6$ *i.e.* $2 : 3$ externally.

28. (c) : It can easily be seen that $AB = BC = CD = DA = 7$. So, $ABCD$ is a rhombus or square.

Also, $AC = \sqrt{122}$; $BD = \sqrt{74}$

$\because$ Diagonals are not equal

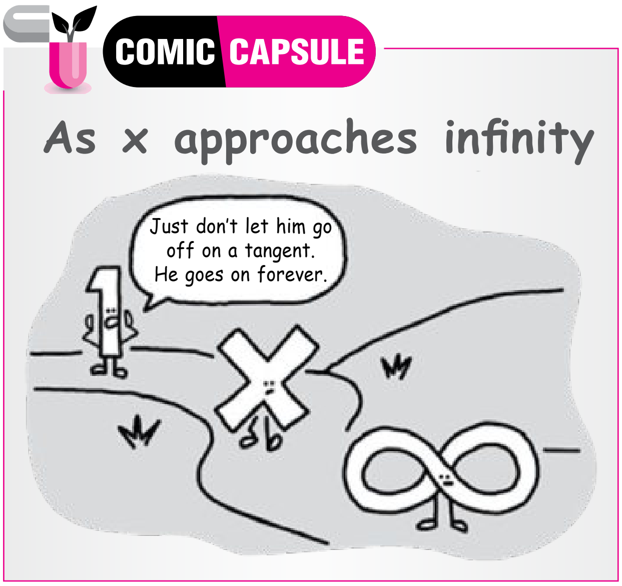
$\therefore$ $ABCD$ is a rhombus

29. (c) : The coordinates of A, B, C are $A(l, -m, n), B(l, m, -n)$, and $C(-l, m, n)$

$$\therefore \quad \frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2} = \frac{4(m^2 + n^2) + 4(l^2 + n^2) + 4(l^2 + m^2)}{l^2 + m^2 + n^2}$$
$$= 8$$

30. (b) : We have, $c = AB = 13$, $b = AC = 5$ and $a = BC = \sqrt{122}$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 169 - 122}{2 \times 5 \times 13} = \frac{36}{65}$$



YOU ASK WE ANSWER

Do you have a question that you just can't get answered?
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. Solve for real x:

$4^x - 3^{(x-1/2)} = 3^{(x+1/2)} - 2^{(2x-1)}$

(Rahul Ganguly, Kolkata)

Ans. We have, $4^x - \frac{3^x}{\sqrt{3}} = 3^x \cdot \sqrt{3} - \frac{4^x}{2}$

$$\Rightarrow \quad 4^x\left(1+\frac{1}{2}\right) = 3^x\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad 4^x\left(\frac{3}{2}\right) = 3^x\left(\frac{3+1}{\sqrt{3}}\right) \Rightarrow \frac{4^x}{3^x} = \frac{8}{3\sqrt{3}}$$

$$\Rightarrow \quad \left(\frac{4}{3}\right)^x = \left(\frac{4}{3}\right)^{3/2} \Rightarrow x - \frac{3}{2} = 0 \Rightarrow x = \frac{3}{2}.$$

2. If S_n represents the sum of the product of the first n natural numbers taken two at a time, then show that $\frac{2}{3!} + \ldots\ldots + \frac{S_{n-1}}{n!} + \ldots\ldots \infty = \frac{11e}{24}$.

(Pooja Shukla, Bihar)

Ans. We have

$(\Sigma n)^2 = \Sigma n^2 + 2S_n$

i.e., $S_n = \frac{1}{2}[(\Sigma n)^2 - \Sigma n^2] = \frac{1}{2}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$

$$= \frac{n(n+1)}{24}[3n(n+1) - 2(2n+1)] = \frac{n(n+1)}{24}[3n^2 - n - 2]$$

$$= \frac{n(n+1)(n-1)(3n+2)}{24}$$

Now, L.H.S. of the given expression is

$$\sum_{n=2}^{\infty} \frac{S_n}{(n+1)!}$$

where $\frac{S_n}{(n+1)!} = \frac{n(n+1)(n-1)(3n+2)}{24(n+1)!} = \frac{3n+2}{24(n-2)!}$

$$= \frac{3(n-2)+8}{24(n-2)!} = \frac{1}{8(n-3)!} + \frac{1}{3(n-2)!}$$

Hence, we have

$$\text{L.H.S.} = \frac{1}{8}\sum_{n=3}^{\infty}\frac{1}{(n-3)!} + \frac{1}{3}\sum_{n=2}^{\infty}\frac{1}{(n-2)!}$$

$$= \left(\frac{1}{8}+\frac{1}{3}\right)e = \frac{11e}{24} = \text{R.H.S.}$$

3. Evaluate the definite integral

$$\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt.$$

(Saaransh Gupta, U.P.)

Ans. We have

$$I = \int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt$$

$$= [t\sin^{-1}(\sqrt{t})]_0^{\sin^2 x} - \int_0^{\sin^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt + [t\cos^{-1}(\sqrt{t})]_0^{\cos^2 x} + \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt$$

$$= x\sin^2 x + \int_{\sin^2 x}^{0} \frac{\sqrt{t}}{2\sqrt{1-t}}dt + x\cos^2 x + \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt$$

$$= x(\sin^2 x + \cos^2 x) + \int_{\sin^2 x}^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt$$

Putting $t = \sin^2\theta$ and $dt = 2\sin\theta\cos\theta\, d\theta$, we have

$$\int \frac{\sqrt{t}}{2\sqrt{1-t}}dt = \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\sin\theta\cos\theta\, d\theta$$

$$= \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

Also, when $t = \sin^2 x$, then $\theta = x$ and when $t = \cos^2 x$, then $\theta = \frac{\pi}{2} - x$.

Hence we have $I = x + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_x^{\pi/2-x}$

$$= x + \left(\frac{\pi}{4} - \frac{x}{2} - \frac{\sin 2x}{4}\right) - \left(\frac{x}{2} - \frac{\sin 2x}{4}\right) = x + \frac{\pi}{4} - x = \frac{\pi}{4}.$$

CONCEPT MAP

LIMITS

Class XI

Some Standard Limits

- $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$, n being a positive integer
- $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$
- $\lim\limits_{x\to a}\dfrac{\sin(x-a)}{x-a}=1$
- $\lim\limits_{x\to 0}\cos x=1$
- $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$
- $\lim\limits_{x\to a}\dfrac{\tan(x-a)}{x-a}=1$
- $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=0$

Limits of Trigonometric Functions

Let f and g be two real valued functions with the same domain such that $f(x)\le g(x)$ for all x in the domain of f and g. For some a, if both $\lim\limits_{x\to a}f(x)$ and $\lim\limits_{x\to a}g(x)$ exist, then $\lim\limits_{x\to a}f(x)\le\lim\limits_{x\to a}g(x)$.

Sandwich Theorem : Let f, g and h be real functions such that $f(x)\le g(x)\le h(x)$ for all $x\in\{\text{dom } f(x)\cap\text{dom } g(x)\cap\text{dom } h(x)\}$. For some real number a, if $\lim\limits_{x\to a}f(x)=l=\lim\limits_{x\to a}h(x)$, then $\lim\limits_{x\to a}g(x)=l$.

Algebra of Limits

Let $\lim\limits_{x\to a}f(x)=l$, $\lim\limits_{x\to a}g(x)=m$. Then

- $\lim\limits_{x\to a}(f(x)\pm g(x))=l\pm m$
- $\lim\limits_{x\to a}kf(x)=kl$
- $\lim\limits_{x\to a}f(x)g(x)=lm$
- $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{l}{m}, m\ne 0$
- $\lim\limits_{x\to a}|f(x)|=|l|$
- $\lim\limits_{x\to a}f(x)^{g(x)}=l^m$
- $\lim\limits_{x\to a}e^{f(x)}=e^l$
- $\lim\limits_{x\to a}(\ln f(x))=\ln l, l>0$

L' Hospital's Rule

Let $f(a)=0$, $g(a)=0$ and $f(x)$, $g(x)$ are differentiable functions with derivatives $f'(x)$, $g'(x)$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{g(x)-g(a)}$

$$=\lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}=\frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}$$

or $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{f'(a)}{g'(a)}=\dfrac{f''(a)}{g''(a)}$ if $f'(a)=g'(a)=0$ and so on.

Definition

Limit of a function $y=f(x)$ exists iff L.H.L = R.H.L. $=f(a)$ where L.H.L $=\lim\limits_{x\to a^-}f(x)$ means limit approaching curve at $x=a$ to the left of a and R.H.L. $=\lim\limits_{x\to a^+}f(x)$ means limit approaching curve at $x=a$ to the right of a.

Limit of a function at a point

Limit of a function at $x=a$ is the value of $f(x)$ at $x=a$ i.e., $\lim\limits_{x\to a}f(x)$.

Limit of a function at an interval

Let $f: R\to R$ be a function and $a\in R$, then we say that $\lim\limits_{x\to a}f(x)=l$ if $|f(x)-l|\to 0$ as $|x-a|\to 0$, where l is some real number.

Evaluation of Limits by Series

Sometimes, $\lim\limits_{x\to a}f(x)$ can be evaluated by using the following series:

- $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+...$
- $a^x=1+(\ln a)\dfrac{x}{1!}+(\ln a)^2\dfrac{x^2}{2!}+(\ln a)^3\dfrac{x^3}{3!}+...$
- $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+...,-1<x<1$
- $\sin x=\dfrac{x}{1!}-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+...$
- $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+...$
- $\tan x=x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+...$
- $x\cot x=1-\dfrac{x^2}{3}-\dfrac{x^4}{45}-....$
- $\sec x=1+\dfrac{x^2}{2}+\dfrac{5}{24}x^4+...$
- $x\operatorname{cosec} x=1+\dfrac{x^2}{6}+\dfrac{7x^4}{360}+...$
- $\sin^{-1}x=x+\dfrac{1}{2}\dfrac{x^3}{3}+\dfrac{1}{2}\dfrac{3}{4}\dfrac{x^5}{5}+\dfrac{1}{2}\dfrac{3}{4}\dfrac{5}{6}\dfrac{x^7}{7}+...$
- $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+...$
- $(1+x)^{1/x}=e^{\left(1-\frac{x}{2}+\frac{11x^2}{24}-...\right)}$

APPLICATION OF DERIVATIVES

Class XII

CONCEPT MAP

Rate of Change of Quantities

- Let $y=f(x)$ then $\dfrac{dy}{dx}$ or $f'(x)$ denotes the rate of change of y w.r.t. x and its value at $x=a$ is denoted as $\left[\dfrac{dy}{dx}\right]_{x=a}$.

Marginal Cost and Marginal Revenue

- Let C be the total cost of producing and marketing x units of a product, then marginal cost (MC), is $MC=\dfrac{dC}{dx}$.
- The rate of change of total revenue with respect to the quantity sold is the marginal revenue, $MR=\dfrac{dR}{dx}$.

Errors and Approximations

Let $y=f(x)$, Δx be the small change in x and Δy be the corresponding change in y. Then, $\Delta y=\dfrac{dy}{dx}(\Delta x)$

These small values Δx and Δy are called differentials.

(i) Absolute Error : Δx

(ii) Relative Error : $\dfrac{\Delta x}{x}$

(iii) Percentage Error : $\left(\dfrac{\Delta x}{x}\times 100\right)$

Increasing and Decreasing Functions

Increasing Function

Increasing Function	without derivative test	If $x_1<x_2\Rightarrow f(x_1)\le f(x_2)\ \forall\ x_1, x_2\in(a,b)$
	with derivative test	If $f'(x)\ge 0$ for each $x\in(a,b)$
Strictly Increasing Function	without derivative test	If $x_1<x_2\Rightarrow f(x_1)<f(x_2)\ \forall\ x_1, x_2\in(a,b)$
	with derivative test	If $f'(x)>0$ for each $x\in(a,b)$

Decreasing Function

Decreasing Function	without derivative test	If $x_1<x_2\Rightarrow f(x_1)\ge f(x_2)\ \forall\ x_1, x_2\in(a,b)$
	with derivative test	If $f'(x)\le 0$ for each $x\in(a,b)$
Strictly Decreasing Function	without derivative test	If $x_1<x_2\Rightarrow f(x_1)>f(x_2)\ \forall\ x_1, x_2\in(a,b)$
	with derivative test	If $f'(x)<0$ for each $x\in(a,b)$

Maxima And Minima

They exist at critical points only.

Critical Points

1. $f(x)$ doesn't exist
2. $f'(x)$ doesn't exist
3. $f'(x)=0$

Global Max., Global Min.

Global Max. : Max. value of $f(x)$ in domain.

Global Min. : Least value of $f(x)$ in domain.

Local Maxima And Local Minima

Local Maxima : At $x=a$

$f(a-h)<f(a)>f(a+h)\ (h\to 0)$

Local Minima : At $x=a$

$f(a-h)>f(a)<f(a+h)\ (h\to 0)$

First derivative test

1. $x=a$ is local max. if $f'(a)=0$ and $f'(x)$ changes from +ve to –ve.
2. $x=a$ is local min. if $f'(a)=0$ and $f'(x)$ changes from –ve to +ve.

Second derivative test

1. Find the roots of $f'(x)=0$
2. Suppose $x=a$ is one of the roots.
3. At those points, if $f''(x)<0\Rightarrow f(x)$ is maximum at $x=a$, or if $f''(x)>0\Rightarrow f(x)$ is minimum at $x=a$.

Higher order derivative test

1. If $f''(a)=0$ check $f'''(x)$ at $x=a$. If $f'''(a)=0$, neither max. nor min.
2. $f''(a)=0$ repeat process considering $f'(x)$ as $g(x)$.

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE Main & Advanced Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE Main & Advanced. In every issue of MT, challenging problems are offered with detailed solution. The reader's comments and suggestions regarding the problems and solutions offered are always welcome.

1. If A is a square matrix such that $A(\text{adj}A)=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$, then $\dfrac{|\text{adj}(\text{adj}A)|}{|\text{adj}A|}$ is equal to

(a) 256 (b) 64 (c) 32 (d) 16

2. If two distinct chords of a parabola $y^2 = 4ax$, passing through $(a, 2a)$ are bisected by the line $x + y = 1$, then length of latus rectum can be

(a) 2 (b) 4 (c) 5 (d) 6

3. The vectors

$(al+a'l')\hat{i}+(am+a'm')\hat{j}+(an+a'n')\hat{k}$,

$(bl+b'l')\hat{i}+(bm+b'm')\hat{j}+(bn+b'n')\hat{k}$,

$(cl+c'l')\hat{i}+(cm+c'm')\hat{j}+(cn+c'n')\hat{k}$

(a) form an equilateral triangle
(b) are coplanar (c) are collinear
(d) are mutually perpendicular

4. $f(x)=\cos x-\int_0^x (x-t)f(t)dt$, then $f''(x) + f(x)$ is equal to

(a) $-\cos x$ (b) $-\sin x$

(c) $\int_0^x (x-t)f(t)dt$ (d) zero

5. If $f(x)=\int_1^x \dfrac{\tan^{-1}t}{t}dt,\ x>0$, then the value of $f(e^2)-f\left(\dfrac{1}{e^2}\right)$ is

(a) 0 (b) $\pi/2$ (c) π (d) 2π

6. The number of solutions of the equation $\tan^{-1}\left(\dfrac{x}{1-x^2}\right)+\tan^{-1}\left(\dfrac{1}{x^3}\right)=\dfrac{3\pi}{4}$, belonging to the interval $(0, 1)$ is

(a) 0 (b) 1 (c) 2 (d) infinite

7. With respect to a variable point on the line $x+y=2a$, chord of contact of the circle $x^2 + y^2 = a^2$ is drawn. If it passes through a fixed point F, the chord of the circle with F as mid point is

(a) parallel to the line $x + y = 2a$
(b) perpendicular to the line $x + y = 2a$
(c) makes angle 45° with the line $x + y = 2a$
(d) none of these

8. The area of the region bounded by the curves $|y + x| \le 1$, $|y - x| \le 1$ and $3x^2 + 3y^2 = 1$ is

(a) $\left(1-\dfrac{\pi}{3}\right)$ sq. units (b) $\left(2-\dfrac{\pi}{3}\right)$ sq. units

(c) $\left(3-\dfrac{\pi}{3}\right)$ sq. units (d) none of these

9. Solution of the differential equation $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{1}{(1+\log x+\log y)^2}$ is

(a) $xy(1+(\ln xy)^2)=\dfrac{x^2}{2}+C$

(b) $xy(1+\ln xy)=\dfrac{x^2}{2}+C$

(c) $xy(1+\ln xy)=\dfrac{x}{2}+C$ (d) none of these

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

10. $P(t^2, 2t)$, $t \in (0, 1]$ is any arbitrary point on $y^2 = 4x$. 'Q' is the foot of perpendicular drawn from focus 'S' to the tangent drawn at P. Maximum area of triangle PQS is

(a) 1 sq. units (b) 2 sq. units
(c) 1/2 sq. units (d) 4 sq. units

SOLUTIONS

1. (d) : $A(\text{adj}A) = |A| \cdot I_n$; Clearly $|A| = 4$, $n = 3$
Now, $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2} = 4^4 = 256$
and $|\text{adj}A| = |A|^{n-1} = 4^2 = 16$.

$$\therefore \frac{|\text{adj (adj } A)|}{|\text{adj } A|} = \frac{256}{16} = 16$$

2. (a) : Any point on the line $x + y = 1$ can be taken as $(t, 1 - t)$
Equation of chord with $(t, 1 - t)$ as mid point is
$y(1 - t) - 2a(x + t) = (1 - t)^2 - 4at$
It passes through $(a, 2a)$ $\therefore$ $t^2 - 2t + 2a^2 - 2a + 1 = 0$
This should have two distinct real roots so $a^2 - a < 0$
$\Rightarrow$ $0 < a < 1 \Rightarrow$ length of latus rectum < 4.

3. (b) :
$$\Delta = \begin{vmatrix} al + a'l' & am + a'm' & an + a'n' \\ bl + b'l' & bm + b'm' & bn + b'n' \\ cl + c'l' & cm + c'm' & cn + c'n' \end{vmatrix}$$

$$= \begin{vmatrix} a & a' & 0 \\ b & b' & 0 \\ c & c' & 0 \end{vmatrix} \times \begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} = 0$$

4. (a) :
$$f'(x) = -\sin x - \left[xf(x) + \int_0^x f(t)dt \right] + xf(x)$$
$$= -\sin x - \int_0^x f(t)\,dt$$
$f''(x) = -\cos x - f(x)$ $\Rightarrow$ $f''(x) + f(x) = -\cos x$

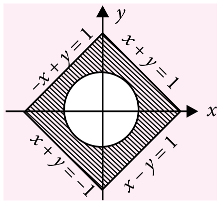
5. (c) :
$$f(x) = \int_1^x \frac{\tan^{-1} t}{t} dt$$
$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\tan^{-1} t}{t} dt = -\int_1^x \frac{\cot^{-1} t}{t} dt$$
$$\Rightarrow \quad f(x) - f\left(\frac{1}{x}\right) = \frac{\pi}{2} \log x$$
$$\Rightarrow \quad f(e^2) - f\left(\frac{1}{e^2}\right) = \frac{\pi}{2} \log_e e^2 = \frac{\pi}{2} \times 2 = \pi$$

6. (a) :
$$\left(\frac{x}{1-x^2}\right) \times \frac{1}{x^3} = \left(\frac{1}{1-x^2}\right) \frac{1}{x^2} > 1$$

$$\text{So, } \tan^{-1}\left(\frac{x}{1-x^2}\right) + \tan^{-1}\left(\frac{1}{x^3}\right)$$
$$= \pi + \tan^{-1}\left(\frac{x^4 + 1 - x^2}{\left(x^2 - x^4 - 1\right)x}\right) = \pi + \tan^{-1}\left(-\frac{1}{x}\right) = \frac{3\pi}{4}$$
$\Rightarrow$ $x = 1$.

7. (a) : Any point on the line $x + y = 2a$ is $(t, 2a - t)$
Now equation of chord of contact is
$xt + y(2a - t) = a^2$ or $(x - y)t + 2ay - a^2 = 0$
This passes through $F\left(\frac{a}{2}, \frac{a}{2}\right)$, now equation of chord with F as mid point is $\frac{xa}{2} + \frac{ya}{2} = \frac{a^2}{2}$
$\Rightarrow$ $x + y = a$, clearly this is parallel to the line $x + y = 2a$.

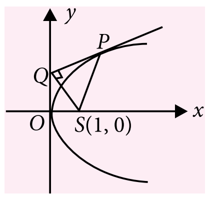
8. (b) : $|y + x| \le 1$, $|y - x| \le 1$ together form a square of side $\sqrt{2}$ units.
$3x^2 + 3y^2 = 1$ is a circle of radius $1/\sqrt{3}$ units.



Now, area of circle $= \pi/3$ sq. units.
and area of square $= 2$ sq. units.
$\therefore$ Required area $= \left(2 - \frac{\pi}{3}\right)$ sq. units.

9. (a) : We have, $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{(1 + \log xy)^2}$
$$\Rightarrow \quad xdy + ydx = \frac{xdx}{(1 + \log(xy))^2}$$
$\Rightarrow$ $(1 + \log xy)^2\, d(xy) = xdx$
Integrating both sides, we get
$$\int (1 + \log t)^2 dt = \frac{x^2}{2} + C \quad (\text{Putting } t = xy)$$
$$\Rightarrow \quad (1 + \log t)^2 t - 2\int (1 + \log t)\, dt = \frac{x^2}{2} + C$$
$$\Rightarrow \quad t(1 + \log t)^2 - 2t - 2(t \log t - t) = \frac{x^2}{2} + C$$
$$\Rightarrow \quad t(1 + \log t)^2 - 2t \log t = \frac{x^2}{2} + C$$
$$\Rightarrow \quad t(1 + (\log t)^2) = \frac{x^2}{2} + C \Rightarrow xy(1 + (\ln xy)^2) = \frac{x^2}{2} + C$$

10. (a) : Equation of PQ is $yt = x + t^2$
$Q \equiv (0, t)$ $\Rightarrow PQ = \sqrt{t^4 + t^2} = t\sqrt{1 + t^2}$
$QS = \sqrt{1 + t^2}$



$$\Rightarrow \quad \Delta PQS = \frac{1}{2} PQ \times QS = \frac{1}{2} t(1 + t^2)$$
which is increasing function of t
$(\Delta PQS)_{\text{max.}} = 1$ sq. units.

◈◈

OLYMPIAD CORNER

1. Solve the simultaneous equations
$x^2(y + z) = 1,\ y^2(z + x) = 8,\ z^2(x + y) = 13$

2. $ABCD$ is a convex quadrilateral, and O is the intersection of its diagonals. Suppose that the area of the (nonconvex) pentagon $ABOCD$ is equal to the area of the triangle OBC. Let P and Q be the points on BC such that $OP \parallel AB$ and $OQ \parallel DC$. Prove that $[OAB] + [OCD] = 2[OPQ]$, where $[XYZ]$ denotes the area of triangle XYZ.

3. In a television commercial some months ago, a pizza restaurant announced a special sale on two pizzas, in which each pizza could independently contain up to five of the toppings the restaurant had available (no topping at all is also an option). In the commercial, a small boy declared that there were a total of 1048576 different possibilities for the two pizzas one could order. How many toppings are available at the restaurant?

4. A fair coin is tossed repeatedly till it shows up heads for the first time. Let n be the number of coin tosses required for this. We then choose at random one of the n integers 1 to n. Find the probability that the chosen integer is 1.

5. Find an integer $n > 1$ so that there exist n consecutive integer squares having an average of n^2.

6. $ABCDE$ is a convex pentagon in the plane. Through each vertex draw a straight line which cuts the pentagon into two parts of the same area. Prove that for some vertex, the line through it must intersect the "opposite side" of the pentagon. (Here the opposite side to vertex A is the side CD, the opposite side to B is DE, and so on).

7. ABC is a triangle which is not equilateral, with circumcentre O and orthocentre H. Point K lies on OH so that O is the midpoint of HK. AK meets BC in X and Y, Z are the feet of the perpendiculars from X onto the sides AC, AB respectively. Prove that AX, BY, CZ are concurrent or parallel.

SOLUTIONS

1. More generally we can replace the constants 1, 8, 13 by a^3, b^3, c^3, respectively. Then by addition of the three equations and by multiplication of the three equations, we respectively get

$\sum x^2y = a^3 + b^3 + c^3$,

$(xyz)^2\ [2xyz + \sum x^2y] = (abc)^3$,

where the sums are symmetric over x, y, z. Hence,

$$2t^3 + t^2(a^3 + b^3 + c^3) = (abc)^3 \quad \ldots(1)$$

where $t = xyz$. In terms of t, the original equations can be rewritten as

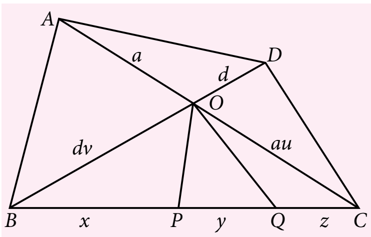
$$\frac{a^3}{tx} - \frac{1}{y} - \frac{1}{z} = 0,\quad \frac{-1}{x} + \frac{b^3}{ty} - \frac{1}{z} = 0,\quad \frac{-1}{x} - \frac{1}{y} + \frac{c^3}{tz} = 0$$

These latter homogeneous equations are consistent since the eliminant is equation (1). Solving the last two equations for y and z, we get

$$y = \frac{x(b_1c_1 - 1)}{c_1 + 1},\quad z = \frac{x(b_1c_1 - 2)}{b_1 + 1}$$

where $b_1 = b^3/t$, $c_1 = c^3/t$. On substituting back in $x^2(y + z) = a^3$, we obtain x^3 and then x, y, z.

2.



Let $[DOC] = u[AOD]$ and $[BOA] = v[AOD]$.
Then, $[BCO] = uv[AOD]$, and the condition of the problem implies

$$1 + u + v = uv \quad ...(1)$$

Furthermore, $\frac{AD}{OC} = \frac{1}{u} = \frac{BP}{PC}$ and $\frac{DO}{OB} = \frac{1}{v} = \frac{CQ}{QB}$.

Let $BP = x$, $PQ = y$ and $QC = z$. Then since $OP \parallel AB$ and $OQ \parallel DC$

$$\therefore \quad \frac{x}{y+z} = \frac{1}{u} \text{ and } \frac{z}{x+y} = \frac{1}{v}$$

so $z = ux - y$ and $x = vz - y$.

We may let $y = 1$, which implies $x = v(ux - 1) - 1$,

so $x = \frac{v+1}{uv-1}$

Similarly, $z = \frac{u+1}{uv-1}$

and thus by (1)

$$x + y + z = \frac{v+1+uv-1+u+1}{uv-1} = \frac{2uv}{u+v}.$$

Now $[BPO] : [PQO] : [QCO] = x : y : z$, and so

$$[OPQ] = \frac{y}{x+y+z}[BCO]$$

$$= \frac{u+v}{2uv}.uv[AOD] = \frac{u+v}{2}[AOD]$$

$= \frac{[DOC]+[BOA]}{2}$, which implies the result.

3. If there are n toppings available at the restaurant then there are

$$k = \binom{n}{5}+\binom{n}{4}+\binom{n}{3}+\binom{n}{2}+\binom{n}{1}+\binom{n}{0}$$

ways to have a single pizza, since you could have up to five toppings (no toppings is a valid, although boring, choice). For two pizzas, you could have two of the same pizza (k ways) or two different pizzas ($\binom{k}{2} = k(k-1)/2$ ways). Since the total number of possible combinations of the two is given as 1048576, the following must be true:

$$\frac{k(k-1)}{2} + k = 1048576$$

But this equation doesn't have any positive integer solutions. It would seem that the restaurant in question considers getting a pizza with bacon and cheese and a pizza with mushrooms, cheese and green pepper different from getting a pizza with mushrooms, cheese and green pepper and a pizza with bacon and cheese; *i.e.*, the order in which the order is placed matters. This seems silly to me, but so be it. In this case the equation that must be solved is $k^2 = 1048576 = 2^{20}$, so $k = 1024 = 2^{10}$. Now to find n we must solve

$$\binom{n}{5}+\binom{n}{4}+\binom{n}{3}+\binom{n}{2}+\binom{n}{1}+\binom{n}{0} = 2^{10}. \quad ...(1)$$

By the symmetry of Pascal's triangle we get

$$\binom{n}{0}=\binom{n}{n}, \binom{n}{1}=\binom{n}{n-1}, \ldots, \binom{n}{5}=\binom{n}{n-5},$$

so our equation (1) can also be written as

$$\binom{n}{n}+\binom{n}{n-1}+\binom{n}{n-2}+\binom{n}{n-3}+\binom{n}{n-4}+\binom{n}{n-5}=2^{10}.$$

If we add these two together we get

$$\binom{n}{n}+\binom{n}{n-1}+\ldots+\binom{n}{n-5}+\binom{n}{5}+\binom{n}{4}+\ldots+\binom{n}{0}$$
$$= 2 \cdot 2^{10} = 2^{11},$$

which is satisfied if $n = 11$, since

$$\sum_{i=0}^{n}\binom{n}{i} = 2^n.$$

Thus the restaurant has 11 different toppings.
If the order in which the order is placed doesn't matter, then the number of two-pizza orders is

$\frac{k(k+1)}{2} = \frac{1024 \cdot 1025}{2} = 524800$, not 1048576.

4. Define the random variable X to be the number of tosses of a fair coin until a head first appears. Then X is a geometric random variable with

$$P(X = n) = \left(\frac{1}{2}\right)^n.$$

The probability that the integer 1 is chosen is given by the formula

$$\sum_{n=1}^{\infty} P(X=n)\frac{1}{n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \frac{1}{n} = \ln 2$$

To see the latter equality, write

$$f(x) = \sum_{n=1}^{\infty} x^n \frac{1}{n}, \quad -1 < x < 1.$$

Then, $f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$.

Therefore $f(x) = -\ln|1-x|$. In particular, $f(1/2) = \ln 2$.
The problem can be generalized to show that the probability $p(m)$ that any positive integer m is chosen is given by the formula

$$\sum_{n=m}^{\infty}\left(\frac{1}{2}\right)^n\frac{1}{n},$$

which is a nice way of showing that

$$\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\left(\frac{1}{2}\right)^n\frac{1}{n}=1$$

5. We seek positive integers k and n, with $n > 1$, such that

$$n^2=\frac{1}{2}\sum_{i=1}^{n}(k+i)^2=\frac{1}{n}\left(\sum_{i=1}^{n}k^2+2k\sum_{i=1}^{n}i+\sum_{i=1}^{n}i^2\right)$$

$$=\frac{1}{n}\left(nk^2+kn(n+1)+\frac{n}{6}(n+1)(2n+1)\right)$$

$$= k^2 + (n + 1)k + \frac{(n+1)(2n+1)}{6} \quad \ldots\text{(i)}$$

Note that n is odd [since otherwise the fraction in (i) is not an integer]

Complete the square in (i) by putting

$$k = a - \frac{n+1}{2},$$

where k is an integer, then (i) becomes

$$n^2 = a^2 - \left(\frac{n+1}{2}\right)^2 + \frac{(n+1)(2n+1)}{6}$$

$$= a^2 + \frac{n+1}{12}[-3(n + 1) + 2(2n + 1)]$$

$$= a^2 + \frac{(n+1)(n-1)}{12}=a^2+\frac{n^2-1}{12},$$

or $12a^2 = 11n^2 + 1$. Putting $n = a + t$ this becomes $12a^2 = 11a^2 + 22at + 11t^2 + 1$

or $(a - 11t)^2 - 132t^2 = 1$.

With $x = a - 11t$ and $y = 2t$ one gets the Pell equation

$$x^2 - 33y^2 = 1. \quad \ldots\text{(ii)}$$

The smallest positive integer solution is $y = 4$, $x = 23$, which implies $t = 2$, $a = 45$, $n = 47$, $k = 21$, and thus a solution to the problem is

$$\frac{22^2+23^2+\ldots+68^2}{47} = 47^2$$

From the theory of Pell equations (and putting $x_1 = 23$, $y_1 = 4$), there are infinitely many positive integer solutions (x_i, y_i) of (ii), given by

$$x_i + y_i\sqrt{33}=(23+4\sqrt{33})^i,$$

or alternately by

$$x_{i+1} = 23x_i + 4\cdot33y_i,\ \ y_{i+1} = 23y_i + 4x_i.$$

The first four solutions and the corresponding values of k and n are

x	y	k	n
23	4	21	47
1057	184	988	2161
48599	8460	45449	99359
2234497	388976	2089688	4568353

6. We denote the area of the polygon $A_1, A_2 \ldots. A_n$ by $[A_1 A_2 \ldots A_n]$ and we put $[ABCDE] = S$. We assume that the straight line through A which bisects the area of the pentagon does not intersect the opposite side CD.

Then either $[ABC] = \frac{1}{2}S$ or $[ADE] > \frac{1}{2}S$.

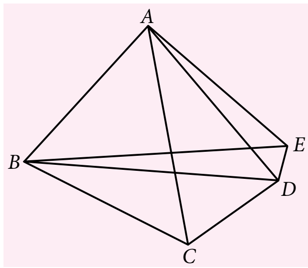
We may assume without loss of generality that

$$[ABC] > \frac{1}{2}S \quad \ldots(1)$$

Now we assume that the straight line through B which bisects the area of the pentagon does not intersect the opposite side DE. Then either $[BAE] > \frac{1}{2}S$ or $[BCD] > \frac{1}{2}S$.

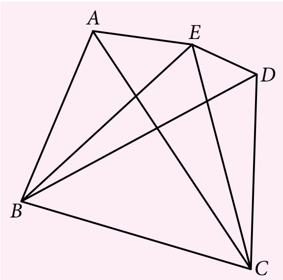
Case 1. $[BAE] > \frac{1}{2}S. \quad \ldots(2)$

From (1) we get that $[ADE] < [ACDE] < \frac{1}{2}S$ and from (2) we get $[BCD] < [BCDE] < \frac{1}{2}S$. Therefore the line through D which bisects the area of the pentagon must intersect the opposite side AB.



Case 2. $[BCD] > \frac{1}{2}S.$ ...(3)

From (1) we get $[ECD] < [ACDE] < \frac{1}{2}S$ and $[EAB] < [EABD] < \frac{1}{2}S$. Therefore the line through E which bisects the area of the pentagon must intersect the opposite side BC.



7. Let $D \in BC$ be the foot of the altitude from A and let M be the midpoint of BC, N the midpoint of AH. Since NH is equal and parallel to OM. $NO \parallel HM$. Since O and N are midpoints of two sides of ΔHKA, $NO \parallel AK$ and so $NO \parallel AX$. Thus $AX \parallel NO \parallel HM$ so that $\Delta ADX \sim \Delta HDM$ and therefore,

$AD : DX = HD : DM$ ...(1)

We now require four facts about triangles with circumradius R, sides a, b, c and angles α, β, γ :

$HD = 2R\cos\beta\cos\gamma$, $DM = R\sin(\beta - \gamma)$, $c = 2R\sin\gamma$, $AD = c\sin\beta = 2R\sin\gamma\sin\beta$.

These can be found or deduced from standard references.

Our ratio (1) is therefore

$2R\sin\gamma\sin\beta : DX = 2\cos\beta\cos\gamma : \sin(\beta - \gamma)$,

so that $DX = \dfrac{R\sin\beta\sin\gamma\sin(\beta-\gamma)}{\cos\beta\cos\gamma}$.

Therefore,

$$MX = DX - R\sin(\beta - \gamma) = R\sin(\beta - \gamma)\left(\frac{\sin\beta\sin\gamma}{\cos\beta\cos\gamma} - 1\right).$$

Finally, after some trigonometry,

$$MX = R\frac{\sin\gamma\cos\gamma - \sin\beta\cos\beta}{\cos\beta\cos\gamma} \quad ...(2)$$

Ceva's theorem says that if

$AZ \cdot BX \cdot CY = ZB \cdot XC \cdot YA,$ ...(3)

then the cevians AX, BY, CZ are concurrent or parallel, which is our goal.

Setting $d = MX$, we find

$AZ = c - (\frac{1}{2}a + d)\cos\beta$, $ZB = (\frac{1}{2}a + d)\cos\beta$,

$BX = \frac{1}{2}a + d$, $XC = \frac{1}{2}a - d$,

$CY = (\frac{1}{2}a - d)\cos\gamma$, $YA = b - (\frac{1}{2}a - d)\cos\gamma$.

Substituting into (3) we find that Ceva's theorem applies if and only if

$$d = \frac{c\cos\gamma - b\cos\beta}{2\cos\beta\cos\gamma} = R\frac{\sin\gamma\cos\gamma - \sin\beta\cos\beta}{\cos\beta\cos\gamma},$$

which agrees with (2) as desired.

CLASS XII **Series 6**

CBSE DRILL

Synopsis and Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2018-19.

Differential Equations

DIFFERENTIAL EQUATION

An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.

Order : Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Degree : Degree of a differential equation, when it is a polynomial equation in derivaties is defined as the highest power (exponent) of the highest order derivative involved in the given differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A relation between the independent and dependent variables free from derivatives satisfying the given differential equation is called a solution of the given differential equation.

General solution : The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.

Particular solution : Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

FORMATION OF A DIFFERENTIAL EQUATION

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

METHODS OF SOLVING DIFFERENTIAL EQUATIONS

Variable Separable Method

If the differential equation is in the form of $\frac{dy}{dx} = \frac{f(x)}{g(x)}$, then the variables are separable and such equations can be solved by integrating on both sides. The solution is given by $\int f(x)dx = \int g(y)dy + C$, where C is an arbitrary constant.

Equations reducible to variable separable : If the given differential equation is of the form $\frac{dy}{dx} = f(ax + by + c)$, put $a\,x + b\,y + c = v$ and it will be reducible to variable separable form.

Homogeneous differential equations

A differential equation of the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$ is said to be homogeneous differential equation if $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree zero.

To solve homogeneous differential equation $\boldsymbol{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$.

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and on substituting these values of y and $\frac{dy}{dx}$ in the given differential equation it will be reducible to variable separable form.

Linear Differential equation

Linear differential equation of first order is of the form $\frac{dy}{dx} + P(x)\cdot y = Q(x)$; where $P(x)$ and $Q(x)$ are functions of x only. To solve this type of equation, first find integrating factor (I.F.) = $e^{\int P\,dx}$ and use the result $y\cdot(\text{I.F.}) = \int(\text{I.F.})\,Q\{x\}dx$ as solution.

Sometimes, we get linear differential equation in the form as $\frac{dx}{dy} + P(y)\cdot x = Q(y)$, where $P(y)$ and $Q(y)$ are functions of y only. In such cases find I.F. = $e^{\int P\,dy}$ and use the result $x\cdot(\text{I.F.}) = \int(\text{I.F.})\;Q(y)\,dy$ as solution.

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Write the order and degree of the differential equation $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0.$
2. Form a differential equation of family of all circles having centres on x-axis and radius unity.
3. Write the integrating factor of the differential equation $(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0.$
4. Solve the differential equation $\frac{dy}{dx} = \frac{x\cdot e^x \log x + e^x}{x\cos y}.$
5. Find the order of the differential equation of the family of curves $y = a\sin(bx + c)$, a and c being parameters.

SHORT ANSWER TYPE

6. Show that $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$
7. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$.
8. A spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.
9. Form the differential equation of the family of ellipses having foci on y-axis and centre at the origin.
10. Find the particular solution of the differential equation $x\frac{dy}{dx} + \frac{y}{\log x} = 1$, given that $y(1) = 1$.

LONG ANSWER TYPE - I

11. If $y(t)$ is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then show that $y(1) = -\frac{1}{2}$.
12. Find the particular solution of the differential equation $y - x\frac{dy}{dx} = a\left(y^2 + x^2\frac{dy}{dx}\right)$, when $x = a, y = a$.
13. Find the equation of the curve passing through the point $(1, -1)$ whose differential equation is $xy\frac{dy}{dx} = (x + 2)(y + 2)$.
14. Find the particular solution of the differential equation $(x - y)(dx + dy) = dx - dy$, given that $y = -1$ when $x = 0$.
15. Find the equation of the curve passing through the point $(0, \pi/4)$ whose differential equation is $\sin x\cos y\,dx + \cos x\sin y\,dy = 0$.

LONG ANSWER TYPE - II

16. Find the general solution of the following differential equation: $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0.$
17. Show that the differential equation $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$.

18. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$ given that $y=\frac{\pi}{2}$ when $x = 1$.

19. Solve : $\frac{dy}{dx}=\frac{1}{\sin^4 x+\cos^4 x}$

20. Solve : $y\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}dx-x\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}dy=0$

SOLUTIONS

1. Highest order derivative is $\frac{dy}{dx}$. Hence, order of given differential equation is 1. Equation cannot be written as a polynomial in derivatives. Hence, degree is not defined.

2. The equation of a circle having centre on x-axis, say at $(a, 0)$, and radius unity is $(x-a)^2+(y-0)^2=1^2$

i.e., $(x-a)^2+y^2=1$, ...(i)

where a is arbitrary real number

Differentiating (i) w.r.t. x, we get

$$2(x-a)\cdot 1+2y\frac{dy}{dx}=0 \Rightarrow x-a=-y\frac{dy}{dx}$$

Substituting this value of $(x-a)$ in (i), we get

$y^2\left(\frac{dy}{dx}\right)^2+y^2=1$, which is the required differential equation.

3. We have, $(1+y^2)+(2xy-\cot y)\frac{dy}{dx}=0$

$$\Rightarrow (2xy-\cot y)\frac{dy}{dx}=-(1+y^2)$$

$$\Rightarrow \frac{dx}{dy}=-\frac{2xy}{1+y^2}+\frac{\cot y}{1+y^2} \Rightarrow \frac{dx}{dy}+\left(\frac{2y}{1+y^2}\right)x=\frac{\cot y}{1+y^2}$$

Now, integrating factor $=e^{\int\frac{2y}{1+y^2}dy}=e^{\log|1+y^2|}=1+y^2.$

4. We have $\frac{dy}{dx}=\frac{xe^x\log x+e^x}{x\cos y}$

$$\Rightarrow \int\cos y\,dy=\int e^x\left(\log x+\frac{1}{x}\right)dx$$

$\Rightarrow \sin y=e^x\log x+C$ is the required solution.

$$\left[\because \int e^x\{f(x)+f'(x)\}dx=e^xf(x)+C\right]$$

5. The equation of the family of curves is

$y=a\sin(bx+c)$ ...(i)

Since, it contains two arbitrary constants. So, we differentiate it two times to get a differential equation of second order.

6. We have, $y=a\cos(\log x)+b\sin(\log x)$ ...(i)

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx}=\frac{-a\sin(\log x)}{x}+\frac{b\cos(\log x)}{x}$$

$$\Rightarrow x\frac{dy}{dx}=-a\sin(\log x)+b\cos(\log x) \quad ...(ii)$$

Differentiating (ii) with respect to x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\frac{a\cos(\log x)}{x}-\frac{b\sin(\log x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-[a\cos(\log x)+b\sin(\log x)]$$

$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0,$ is the differential equation.

Hence, $y=a\cos(\log x)+b\sin(\log x)$ is a solution of the given differential equation.

7. The given equation is $\frac{dy}{dx}=1+x+y+xy$

$$\Rightarrow \frac{dy}{dx}=(1+x)(1+y) \Rightarrow \frac{dy}{1+y}=(1+x)dx$$

Integrating both sides, we get

$$\int\frac{dy}{1+y}=\int(1+x)\,dx \Rightarrow \log|1+y|=x+\frac{x^2}{2}+C \quad ...(i)$$

Given $y=0$, when $x=1$

$\therefore$ From (i), $\log|1+0|=1+\frac{1}{2}+C \Rightarrow C=-\frac{3}{2}$

Substituting the value of C in (i), we get

$\log|1+y|=x+\frac{x^2}{2}-\frac{3}{2}$ which is the required solution.

8. Let r denote the radius of the rain drop after t seconds, since the radius of the rain drop is decreasing as t increases, the rate of change of r is negative.

Let V be the volume of the rain drop and S be its surface area at time t seconds.

$\therefore V=\frac{4}{3}\pi r^3$ and $S=4\pi r^2$

According to question, we have

$$\frac{dV}{dt}\propto S$$

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)=-k\cdot 4\pi r^2,$$

where k is some +ve real number

$$\Rightarrow \frac{4}{3}\pi.3r^2\frac{dr}{dt} = -k.4\pi r^2 \Rightarrow \frac{dr}{dt} = -k,$$

which is the required differential equation.

9. The equation of the family of ellipses having centre at the origin and foci on y-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b > a \quad \text{...(i)}$$

Differentiating (i) with respect to x, we get

$$\Rightarrow \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \quad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\left(\frac{d^2y}{dx^2}\right) = 0 \quad \text{...(iii)}$$

Solving (ii) and (iii), we get

$x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0,$ which is the required differential equation.

10. The given equation is $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{1}{x}$

Now, $\text{I.F.} = e^{\int \frac{1}{x\log x}dx} = e^{\log|\log x|} = |\log x|$

$\therefore$ Solution is, $\log x \cdot y = \int \frac{|\log x|}{x}dx$

$$\Rightarrow \log x \cdot y = \frac{(\log|x|)^2}{2} + C \quad \text{...(i)}$$

Given when $x = 1, y = 1$ $\therefore$ From (i), $C = 0$

Hence, $y = \frac{1}{2}\log|x|$, is the required particular solution.

11. The given equation is $(1+t)\frac{dy}{dt} - ty = 1$

$$\Rightarrow \frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$$

Now, $\text{I.F.} = e^{-\int \frac{t}{1+t}dt} = e^{-\int\left(1-\frac{1}{1+t}\right)dt}$

$$= e^{-t+\log(1+t)} = e^{-t}\cdot e^{\log(1+t)} = (1+t)e^{-t}$$

Hence the solution is

$$(1+t)e^{-t}\cdot y = \int (1+t)\cdot e^{-t}\cdot\frac{1}{1+t}dt = \int e^{-t}dt$$

$$\Rightarrow (1+t)e^{-t}\cdot y = -e^{-t} + C \Rightarrow (1+t)y = -1 + C\cdot e^{t} \quad \text{...(i)}$$

Given, when $t = 0, y = -1$

$\therefore$ From (i), $-1 = -1 + C \Rightarrow C = 0$

$\therefore$ (i) becomes $(1+t)y = -1$

Further, when $t = 1$, we have, $(1+1)y = -1 \Rightarrow y(1) = -\frac{1}{2}$.

12. The given differential equation can be written as

$$y-ay^2=(x+ax^2)\frac{dy}{dx} \Rightarrow \frac{dy}{y(1-ay)}=\frac{dx}{x(1+ax)}$$

$$\Rightarrow \int\frac{dy}{y(1-ay)}=\int\frac{dx}{x(1+ax)}$$

$$\Rightarrow \int\frac{(1-ay)+ay}{y(1-ay)}dy=\int\frac{(1+ax)-ax}{x(1+ax)}dx$$

$$\Rightarrow \int\frac{1}{y}dy+a\int\frac{1}{1-ay}dy=\int\frac{1}{x}dx-a\int\frac{1}{1+ax}dx$$

$$\Rightarrow \log|y|+a\cdot\frac{\log|1-ay|}{-a}=\log|x|-a\cdot\frac{\log|1+ax|}{a}+C$$

$\Rightarrow$ $\log|y| - \log|1 - ay| = \log|x| - \log|1 + ax| + C$ ...(i)

Given, when $x = a$, $y = a$

$\Rightarrow$ $\log a - \log|1 - a^2| = \log a - \log(1 + a^2) + C$

$$\Rightarrow C=\log\left|\frac{1+a^2}{1-a^2}\right|$$

Substituting the value of C in (i), we get

$$\log|y|-\log|1-ay|=\log|x|-\log|1+ax|+\log\left|\frac{1+a^2}{1-a^2}\right|$$

$$\Rightarrow \log\left|\frac{y}{1-ay}\right|=\log\left|\frac{x(1+a^2)}{(1+ax)(1-a^2)}\right|$$

$\Rightarrow$ $y(1 + ax)(1 - a^2) = x(1 + a^2)(1 - ay)$, is the required solution.

13. Given $xy\frac{dy}{dx} = (x + 2)(y + 2)$ ...(i)

$$\Rightarrow \frac{y}{y+2}dy=\frac{x+2}{x}dx,\ y\neq-2, x\neq0$$

$$\Rightarrow \left(1-\frac{2}{y+2}\right)dy=\left(1+\frac{2}{x}\right)dx$$

Integrating both sides, we get

$y - 2\log|y + 2| = x + 2\log|x| + C$ ...(ii)

Now, we have when $x = 1$, $y = -1$.

$\therefore$ From (ii), we have $-1 - 2\log 1 = 1 + 2\log 1 + C$

$\Rightarrow$ $C = -2$

Substituting the value of C in (ii), the equation of the required curve is $y - 2\log|y + 2| = x + 2\log|x| - 2$.

14. Given $(x - y)(dx + dy) = dx - dy$

$\Rightarrow$ $(x - y - 1)\,dx + (x - y + 1)\,dy = 0$

$$\Rightarrow \frac{dy}{dx}+\frac{x-y-1}{x-y+1}=0 \quad \text{...(i)}$$

Put $x-y=v \Rightarrow 1-\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=1-\frac{dv}{dx}$

$$\therefore 1-\frac{dv}{dx}+\frac{v-1}{v+1}=0 \Rightarrow \frac{dv}{dx}=1+\frac{v-1}{v+1}=\frac{2v}{v+1}$$

$$\Rightarrow \frac{v+1}{v}dv=2\,dx \Rightarrow \left(1+\frac{1}{v}\right)dv=2dx$$

On integrating both sides, we get

$v + \log|v| = 2x + c \Rightarrow x - y + \log|x - y| = 2x + C$

$\Rightarrow$ $\log|x - y| = x + y + C$ ...(ii)

Now when $x = 0$, $y = -1$

$\Rightarrow$ $\log 1 = 0 - 1 + C \Rightarrow C = 1$.

Substituting the value of C in (ii), we get

$\log|x - y| = x + y + 1$, which is the required particular solution.

15. We have, $\sin x\cos y\,dx + \cos x\sin y\,dy = 0$

$\Rightarrow$ $\tan x\,dx + \tan y\,dy = 0 \Rightarrow \int\tan x\,dx+\int\tan y\,dy=0$

$\Rightarrow$ $-\log|\cos x| - \log|\cos y| = \log C$

$\Rightarrow$ $-\log(|\cos x|\,|\cos y|) = \log C$

$$\Rightarrow \log|\cos x\cos y|=\log\left(\frac{1}{C}\right)$$

$$\Rightarrow |\cos x\cos y|=\left(\frac{1}{C}\right)$$

$\Rightarrow$ $\cos x\cos y = C_1$ where $C_1 = \pm\frac{1}{C}$ ...(i)

It is given that the curve passes through $(0, \pi/4)$.

$$\therefore \cos 0\cos\frac{\pi}{4}=C_1 \Rightarrow C_1=\frac{1}{\sqrt{2}}$$

Putting $C_1=\frac{1}{\sqrt{2}}$ in (i), we get

$$\cos x\cos y=\frac{1}{\sqrt{2}} \Rightarrow \cos y=\frac{1}{\sqrt{2}}\sec x$$

$\Rightarrow$ $y=\cos^{-1}\left(\frac{1}{\sqrt{2}}\sec x\right)$ is the required curve.

16. The given differential equation is,

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$

$$\Rightarrow (x-e^{\tan^{-1}y})\frac{dy}{dx}=-(1+y^2)$$

$$\Rightarrow \frac{-(x-e^{\tan^{-1}y})}{1+y^2}=\frac{dx}{dy} \Rightarrow \frac{dx}{dy}=-\frac{x}{1+y^2}+\frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\therefore \text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

So, the required solution is,

$$e^{\tan^{-1}y}x=\int e^{\tan^{-1}y}\cdot\frac{e^{\tan^{-1}y}}{1+y^2}dy=\int\frac{e^{2\tan^{-1}y}}{1+y^2}dy$$

$$\Rightarrow\ e^{\tan^{-1}y}x=\int e^{2t}dt\left[\text{Let }\tan^{-1}y=t\Rightarrow\frac{1}{1+y^2}dy=dt\right]$$

$$=\frac{e^{2t}}{2}+C$$

$$\Rightarrow\ e^{\tan^{-1}y}x=\frac{1}{2}e^{2\tan^{-1}y}+C\Rightarrow x=\frac{1}{2}e^{\tan^{-1}y}+Ce^{-\tan^{-1}y}$$

17. The given equation is,

$$\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+x\,dy=0$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{y-x\sin^2\left(\frac{y}{x}\right)}{x}\qquad\text{...(i)}$$

Consider $F(x, y)=\dfrac{y-x\sin^2\left(\frac{y}{x}\right)}{x}$

Now, $F(\lambda x,\lambda y)=\dfrac{\lambda y-\lambda x\sin^2\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}=\dfrac{y-x\sin^2\left(\frac{y}{x}\right)}{x}$

$\Rightarrow\ F(\lambda x,\lambda y)=F(x, y)$

Hence, function is homogeneous, so corresponding differential equation is homogeneous.

Let $y = vx\ \Rightarrow\ \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

Now, from (i), we have

$$v+x\frac{dv}{dx}=\frac{vx-x\sin^2 v}{x}=v-\sin^2 v$$

$$\Rightarrow\ x\frac{dv}{dx}=-\sin^2 v\ \Rightarrow\ \frac{dv}{\sin^2 v}=-\frac{dx}{x}$$

$$\Rightarrow\ \int\operatorname{cosec}^2 v\,dv=-\int\frac{dx}{x}+C\ \Rightarrow -\cot v=-\log|x|+C$$

$$\Rightarrow\ \cot\left(\frac{y}{x}\right)=\log|x|-C\qquad\text{...(ii)}$$

It is given that $y=\dfrac{\pi}{4}$, when $x=1$

$$\therefore\ \cot\left(\frac{\pi}{4}\right)=\log|1|-C\ \Rightarrow\ C=-1$$

Substituting the value of C in (ii), we get

$\cot\left(\dfrac{y}{x}\right)=\log|x|+1,$ which is the particular solution.

JEE Main 2019 merit list to be based on percentile scores

From 2019, a JEE-Main aspirant will first time have the choice of taking the test twice (January and April cycles). The format will see the test being conducted over a period of around 14 days in each cycle with multiple sessions each day, a device that is intended to significantly reduce the prospects of cheating and manipulation. Candidates will get different sets of questions per session and though efforts will be made to maintain equivalence among various question papers, the difficulty level of question papers in different sessions may not exactly be same. However, the percentile score of each session is based on the relative performance of students in that particular shift. So the percentile scores of each session will be considered at par.

The JEE (Main)-2019 will be conducted twice before admissions in the next academic session.

What is the new way of ranking in chronological order

1. Each session will have its own NTA percentile score with the highest scorer(s) as 100 percentile
2. A master NTA score will be prepared where all the sessions will be put together for a final ranking
3. The top ranked candidate in the final ranking will be the 100 percentile
4. The tie-breaker between the 100 percentile scorers (or any percentile rank) will be the highest percentile in Mathematics, Physics & Chemistry in that order
5. If scores are tied even after this, the older person will be ranked higher
6. The same method will be applied to the Main 2 exam
7. A final merit list will be announced after Main 2
8. For those who take both cycles of the exam, the better percentile will count in the final merit list.

The students will have following benefits of the new pattern:

- This will give one more opportunity to the students to improve their scores in examination if they fail to give their best in first attempt without wasting their whole academic year.
- In first attempt, the students will get a first-hand experience of taking an examination and to know their mistakes which they can improve while attempting for the second time.
- This will reduce chances of dropping a year and droppers would not have to waste a full year.
- If anyone missed the examination due to reasons beyond control, then he/she won't have to wait for one full year.
- The student's best of the two NTA scores will be considered for preparation of Merit List/ Ranking.

Some candidates may end up attempting a relatively tougher set of questions and are likely to get lower marks. But to ensure a level playing field, 'normalisation procedure based on percentile score' will be used so that candidates are neither benefitted or disadvantaged due to the difficulty level of the examination.

The whole process is divided into three steps, distribution of examinees in the sessions randomly so that each session gets approximately equal number of candidates, preparation of results for each session and finally compilation of the NTA score and preparation of the overall merit/rank list.

18. The given equation is $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$

$\Rightarrow\ (\sin y+y\cos y)dy=x(2\log x+1)dx$

Integrating both sides, we get

$\int(\sin y+y\cos y)dy=\int x(2\log x+1)dx$

$\Rightarrow\ \int\sin y\ dy+\int y\cos y dy\ =2\int x\log x\ dx+\int x\ dx$

$\Rightarrow\ \int\sin y\ dy+y\sin y-\int 1\cdot\sin y\ dy$

$$=2\left[\log x\cdot\frac{x^2}{2}-\int\frac{1}{x}\frac{x^2}{2}dx\right]+\frac{x^2}{2}$$

$\Rightarrow\ y\sin y=x^2\log x-\frac{x^2}{2}+\frac{x^2}{2}+C$

$\Rightarrow\ y\sin y=x^2\log x+C$ ...(i)

It is given that $y=\frac{\pi}{2}$, when $x=1$

$\therefore$ From (i), we have $\frac{\pi}{2}\sin\frac{\pi}{2}=1\cdot\log 1+C\ \Rightarrow\ C=\frac{\pi}{2}$

Substituting the value of C in (i), we get

$y\sin y=x^2\log x+\frac{\pi}{2}$, which is the required solution.

19. We have, $\frac{dy}{dx}=\frac{1}{\sin^4 x+\cos^4 x}$

$\Rightarrow\ dy=\frac{1}{\sin^4 x+\cos^4 x}dx$

Integrating both sides, we get

$\int dy=\int\frac{1}{\sin^4 x+\cos^4 x}dx\ \Rightarrow\ \int dy=\int\frac{\sec^4 x}{\tan^4 x+1}dx$

$\Rightarrow\ \int dy=\int\frac{\sec^2 x\cdot\sec^2 x\,dx}{\tan^4 x+1}$

$\Rightarrow\ \int dy=\int\frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x+1}dx$

$\Rightarrow\ \int dy=\int\frac{1+t^2}{1+t^4}dt$, where $t=\tan x$

$\Rightarrow\ \int dy=\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt\ \Rightarrow\ \int dy=\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$

$\Rightarrow\ \int dy=\int\frac{du}{u^2+(\sqrt{2})^2}$, where $t-\frac{1}{t}=u$

$\Rightarrow\ y=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$

$\Rightarrow\ y=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+C$

$\Rightarrow\ y=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{2}}\right)+C$, which is the required solution of the given differential equation.

20. The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}}\quad\text{...(i)}$$

Putting $y=vx\ \Rightarrow\ \frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v+x\frac{dv}{dx}=\frac{vx\{x\cos v+vx\sin v\}}{x\{vx\sin v-x\cos v\}}=\frac{v\{\cos v+v\sin v\}}{\{v\sin v-\cos v\}}$$

$\Rightarrow\ x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v-v^2\sin v+v\cos v}{v\sin v-\cos v}$

$$=\frac{2v\cos v}{v\sin v-\cos v}$$

$\Rightarrow\ -\int\frac{\cos v-v\sin v}{v\cos v}dv=2\int\frac{dx}{x}$

$\Rightarrow\ -\log|v\cos v|=2\log|x|+\log C$

$\Rightarrow\ \log\frac{1}{|v\cos v|}=\log|x^2|+\log C$

$\Rightarrow\ \left|\frac{1}{v\cos v}\right|=|C|x^2\ \Rightarrow\ \left|\frac{x}{y}\sec\left(\frac{y}{x}\right)\right|=|C|x^2$

$\Rightarrow\ |xy\cos(y/x)|=\frac{1}{|C|}$

$\Rightarrow\ |xy\cos(y/x)|=k$, where $k=1/|C|$

Hence, $\left|xy\cos\left(\frac{y}{x}\right)\right|=k, x\neq 0, k>0$ is the required solution.

MPP-6 CLASS XII ANSWER KEY

1.	(b)	**2.**	(b)	**3.**	(b)	**4.**	(c)	**5.**	(c)
6.	(d)	**7.**	(b,c)	**8.**	(a,d)	**9.**	(a,b,c,d)		
10.	(a,c)	**11.**	(d)	**12.**	(a,c)	**13.**	(b, d)	**14.**	(a)
15.	(b)	**16.**	(a)	**17.**	(124)	**18.**	(7)	**19.**	(0)
20.	(4)								

MPP-6 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Indefinite Integration

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If $I=\int \frac{dx}{x^3\sqrt{x^2-1}}$, then I equals to

(a) $\frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x}+\tan^{-1}\sqrt{x^2-1}\right)+C$

(b) $\frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^2}+\tan^{-1}\sqrt{x^2-1}\right)+C$

(c) $\frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^3}+\tan^{-1}\sqrt{x^2-1}\right)+C$

(d) $\frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^2}+x\tan^{-1}\sqrt{x^2-1}\right)+C$

2. Let $f(x)=\int \frac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0)$ = 0, then $f(1)$ =

(a) $\log(1+\sqrt{2})$ (b) $\log(1+\sqrt{2})-\frac{\pi}{4}$

(c) $\log(1+\sqrt{2})+\frac{\pi}{2}$ (d) None of these

3. $\int \frac{x^2(1-\ln x)}{(\ln x)^4-x^4}dx$ is equal to

(a) $\frac{1}{2}\ln\left(\frac{x}{\ln x}\right)-\frac{1}{4}\ln(\ln^2 x-x^2)+C$

(b) $\frac{1}{4}\ln\left(\frac{\ln x-x}{\ln x+x}\right)-\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$

(c) $\frac{1}{4}\ln\left(\frac{\ln x+x}{\ln x-x}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$

(d) $\frac{1}{4}\ln\left(\frac{\ln x-x}{\ln x+x}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$

4. For any natural number m, $\int (x^{7m}+x^{2m}+x^m)(2x^{6m}+7x^m+14)^{1/m}dx$, where $x>0$ equals

(a) $\frac{(7x^{7m}+2x^{2m}+14x^m)^{(m+1)/m}}{14(m+1)}+C$

(b) $\frac{(2x^{7m}+14x^{2m}+7x^m)^{(m+1)/m}}{14(m+1)}+C$

(c) $\frac{(2x^{7m}+7x^{2m}+14x^m)^{(m+1)/m}}{14(m+1)}+C$

(d) $\frac{(7x^{7m}+2x^{2m}+x^m)^{(m+1)/m}}{14(m+1)}+C$

5. If $I=\int \frac{\tan x}{\sqrt{a+b\tan^2 x}}dx (a>b)$, then I equals

(a) $\frac{1}{\sqrt{b-a}}\sin^{-1}\left(\sqrt{\frac{a+b\tan^2 x}{b-a}}\right)+C$

(b) $\frac{1}{\sqrt{b-a}}\cos^{-1}\left(\sqrt{\frac{a+b\tan^2 x}{b-a}}\right)+C$

(c) $\frac{1}{\sqrt{b-a}}\tan^{-1}\left(\sqrt{\frac{a+b\tan^2 x}{b-a}}\right)+C$

(d) $\frac{1}{\sqrt{b-a}}\tan^{-1}\left(\sqrt{\frac{b-a}{a+b\tan^2 x}}\right)+C$

6. If $f(x)=\frac{x+2}{2x+3}$. Then, $\int \left\{\frac{f(x)}{x^2}\right\}^{\frac{1}{2}}dx$ is equal to

$\frac{1}{\sqrt{2}}g\left\{\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right\}-\sqrt{\frac{2}{3}}h\left\{\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right\}+C$, where

(a) $g(x)=\tan^{-1}x$, $h(x)=\log|x|$
(b) $g(x)=\log|x|$, $h(x)=\tan^{-1}x$
(c) $g(x)=h(x)=\tan^{-1}x$
(d) $g(x)=\log|x|$, $h(x)=\log|x|$

One or More Than One Option(s) Correct Type

7. If $I=\int\log(\sqrt{x-a}+\sqrt{x-b})dx$, then I equals

(a) $[2x-(a+b)\log(\sqrt{x-a}+\sqrt{x-b})]+C$

(b) $\frac{1}{2}[2x-(a+b)](\log(b-a)-\log(\sqrt{x-a}-\sqrt{x-b}))$
$-\left(\frac{1}{2}\right)\sqrt{(x-a)(x-b)}+C$

(c) $\frac{1}{2}[2x-(a+b)]\log(\sqrt{x-a}+\sqrt{x-b})$
$-\frac{1}{2}\sqrt{(x-a)(x-b)}+C$

(d) None of these

8. If $\int\sin^{-1}x\cos^{-1}x\,dx=f^{-1}(x)\times$
$\left[Ax-xf^{-1}(x)-2\sqrt{1-x^2}\right]+A\sqrt{1-x^2}+2x+C$, then

(a) $f(x)=\sin x$ (b) $f(x)=\cos x$

(c) $A=\pi/4$ (d) $A=\pi/2$

9. If $\int\frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx=\frac{A}{2}\sqrt[3]{x^2}+B\tan^{-1}\sqrt[6]{x}+C,$
then

(a) $A+B=9$ (b) $A=3$

(c) $B=6$ (d) $B-A=3$

10. If $\int xe^{-5x^2}\sin 4x^2dx=Ke^{-5x^2}(A\sin 4x^2+B\cos 4x^2)+C,$
then

(a) $K=-1/82$ (b) $K=1/82$

(c) $A=5$ (d) none of these

11. If $I=\int\frac{\sin^3(\theta/2)}{\cos(\theta/2)\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta$, then I equals

(a) $\cot^{-1}(\tan\theta+\sec\theta)+C$

(b) $\cot^{-1}(\cos\theta+\sin\theta+1)+C$

(c) $\tan^{-1}\left(\tan\frac{\theta}{2}+\sec\frac{\theta}{2}+1\right)+C$

(d) $\tan^{-1}(\cos\theta+\sec\theta+1)+C$

12. If $\int\frac{(2x-1)dx}{x^4-2x^3+x+1}=A\tan^{-1}\left[\frac{f(x)}{\sqrt{3}}\right]+C$, then

(a) $A=\frac{2}{\sqrt{3}}$ (b) $A=2$

(c) $f(x)=2x^2-2x-1$ (d) $f(x)=2x^2+2x+1$

13. If $\int\sqrt{\operatorname{cosec}x+1}\,dx=kfog(x)+c$, where k is a real constant, then

(a) $k=-2, f(x)=\cot^{-1}x, g(x)=\sqrt{\operatorname{cosec}x-1}$

(b) $k=-2, f(x)=\tan^{-1}x, g(x)=\sqrt{\operatorname{cosec}x-1}$

(c) $k=2, f(x)=\tan^{-1}x, g(x)=\frac{\cot x}{\sqrt{\operatorname{cosec}x-1}}$

(d) $k=2, f(x)=\cot^{-1}x, g(x)=\frac{\cot x}{\sqrt{\operatorname{cosec}x+1}}$

Comprehension Type

If A is square matrix and e^A is defined as $e^A=I+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots=\frac{1}{4}\begin{bmatrix}f(x) & g(x)\\ g(x) & f(x)\end{bmatrix}$, where $A=\begin{bmatrix}x & x\\ x & x\end{bmatrix}$ and $0<x<1$, I is an identity matrix.

14. $\int\frac{g(x)}{f(x)}dx$ is equal to

(a) $\log(e^x+e^{-x})+c$ (b) $\log|e^x-e^{-x}|+c$

(c) $\log|e^{2x}-1|+c$ (d) None of these

15. $\int(g(x)+1)\sin x\,dx$ is equal to

(a) $\frac{e^x}{2}(\sin x-\cos x)$ (b) $\frac{e^{2x}}{5}(2\sin x-\cos x)$

(c) $\frac{e^x}{5}(\sin 2x-\cos 2x)$ (d) None of these

Matrix Match Type

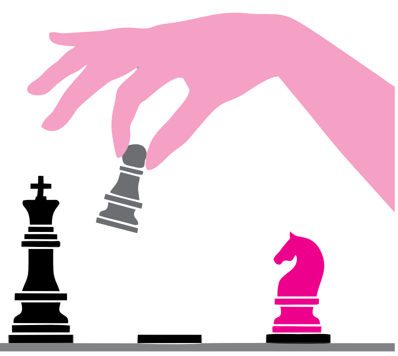
16. Match the following :

	Column-I		Column-II
P.	If $\int\frac{2^x}{\sqrt{1-4^x}}dx=(\log k)^{-1}\sin^{-1}(f(x))+C$, then k is greater than	1.	0
Q.	If $\int\frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}dx=a\ln\frac{x^k}{x^k+1}+c$, then ak is less than	2.	1
R.	If $\int\frac{x^4+1}{x(x^2+1)^2}dx=k\ln\lvert x\rvert+\frac{m}{1+x^2}+n$, where n is the constant of integration, then mk is greater than	3.	3
S.	If $\int\frac{dx}{5+4\cos x}=k\tan^{-1}\left(m\tan\frac{x}{2}\right)+C$, then k/m is greater than	4.	4

Contd. on page no. 83

Challenging PROBLEMS

Vectors and 3D Geometry



1. A plane intersects a tetrahedron *ABCD* and divides the medians of the triangles *DAB*, *DBC* and *DCA* through *D* in ratios 1 : 3, 1 : 4 and 1 : 5 from *D* respectively. The ratio of the volumes of the two parts of the tetrahedron cut by the plane is

(a) $\frac{13}{8}$ (b) $\frac{4}{8}$ (c) $\frac{2}{48}$ (d) $\frac{1}{47}$

2. Let *ABCD* be a tetrahedron and let *E*, *F*, *G*, *H*, *K*, *L* be points lying on the edges *AB*, *BC*, *CA*, *DA*, *DB*, *DC* respectively. In such a way that $AE \cdot BE = BF \cdot CF = CG \cdot AG = DH \cdot AH = DK \cdot BK = DL \cdot CL$, then the points *E*, *F*, *G*, *H*, *K*, *L* lie on a

(a) cube (b) cone
(c) cylinder (d) sphere

3. The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\vec{0}$, then $\vec{a}+\vec{b}+\vec{c}=$

(a) $-\vec{a}$ (b) $-\vec{b}$ (c) $-\vec{c}$ (d) $\vec{0}$

4. Given three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$.

Define $\vec{u}=(\vec{b}\cdot\vec{c})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}$, $\vec{v}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$, $\vec{w}=(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}$. If $\vec{a}$, $\vec{b}$, $\vec{c}$ form a triangle, then

(a) $\vec{u}$, $\vec{v}$, $\vec{w}$ also form a similar triangle with $\vec{a}$, $\vec{b}$, $\vec{c}$.
(b) $\vec{u}$, $\vec{v}$, $\vec{w}$ also form a congruent triangle with $\vec{a}$, $\vec{b}$, $\vec{c}$.
(c) $\vec{u}$, $\vec{v}$, $\vec{w}$ forms an isosceles triangle.
(d) $\vec{u}$, $\vec{v}$, $\vec{w}$ does not form a triangle.

5. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. Then the co-ordinates of the vertex A_1, if the co-ordinates of the base vertices of the prism are *A*(1, 0, 1), *B*(2, 0, 0) and *C*(0, 1, 0) is

(a) (–2, 2, 2) or (0, –2, 1)
(b) (2, 2, 2) or (0, –2, 0)
(c) (0, 2, 0) or (1, –2, 0)
(d) (3, –2, 0) or (1, –2, 0)

6. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three unit vectors such that $\vec{a}+\vec{b}+\vec{c}$ is also a unit vector and θ_1, θ_2 and θ_3 are angles between the vectors $\vec{a},\vec{b};\vec{b},\vec{c}$ and $\vec{c},\vec{a}$, respectively, then among θ_1, θ_2 and θ_3

(a) all are acute angles (b) all are right angles
(c) at least one is obtuse angle
(d) none of these

7. *A*, *B*, *C* are the points on *x*, *y* and *z* axes respectively in a three dimensional co-ordinate system with *O* as origin. Suppose the area of triangles *OAB*, *OBC* and *OCA* are 4, 12 and 6 respectively, then the area of the triangle *ABC* equals

(a) 16 (b) 14 (c) 28 (d) 32

8. Three straight lines mutually perpendicular to each other meet at a point *P* and one of them intersects the *x*-axis and another intersects the *y*-axis, while the third line passes through a fixed point (0, 0, *c*) on the *z*-axis, then the locus of *P* is

(a) $x^2+y^2+z^2-2cx=0$
(b) $x^2+y^2+z^2-2cy=0$
(c) $x^2+y^2+z^2-2cz=0$
(d) $x^2+y^2+z^2-2c(x+y+z)=0$

9. Let *ABCD* be a tetrahedron in which position vectors of *A*, *B*, *C* and *D* are $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+\hat{j}+2\hat{k}$, $3\hat{i}+2\hat{j}+\hat{k}$, and $2\hat{i}+3\hat{j}+2\hat{k}$. If *ABC* be the base of tetrahedron, then height of tetrahedron is

(a) $\sqrt{\frac{3}{2}}$ (b) $\sqrt{\frac{3}{5}}$ (c) $\frac{2\sqrt{2}}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

10. Through a point *P*(*h*, *k*, *l*) a plane is drawn at right angles to *OP* to meet the co-ordinate axes in *A*, *B* and *C*. If *OP* = *p*, then the area of ΔABC is

(a) $\frac{p^2hk}{l^2}$ (b) $\frac{p^3l}{3hk}$ (c) $\frac{p^2l^2}{2hk}$ (d) $\frac{p^5}{2hkl}$

11. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three unit vectors equally inclined to each other at an angle α. Then the angle between $\vec{a}$ and plane of $\vec{b}$ and $\vec{c}$ is

By : Tapas Kr. Yogi, Visakhapatnam Mob : 09533632105

(a) $\theta=\cos^{-1}\left(\dfrac{\cos\alpha}{\cos\dfrac{\alpha}{2}}\right)$ (b) $\theta=\sin^{-1}\left(\dfrac{\cos\alpha}{\cos\dfrac{\alpha}{2}}\right)$

(c) $\theta=\cos^{-1}\left(\dfrac{\sin\dfrac{\alpha}{2}}{\sin\alpha}\right)$ (d) $\theta=\sin^{-1}\left(\dfrac{\sin\dfrac{\alpha}{2}}{\sin\alpha}\right)$

12. If $\vec{a}$ and $\vec{b}$ are unit vectors and $\vec{c}$ is a vector such that $\vec{c}=\vec{a}\times\vec{c}+\vec{b}$, then

(a) $2[\vec{a}\ \vec{b}\ \vec{c}]=\vec{b}\cdot\vec{c}-(\vec{a}\cdot\vec{b})^2$

(b) $[\vec{a}\ \vec{b}\ \vec{c}]=0$

(c) Maximum value of $[\vec{a}\ \vec{b}\ \vec{c}]=\dfrac{1}{2}$

(d) Minimum value of $[\vec{a}\ \vec{b}\ \vec{c}]$ is $\dfrac{1}{2}$

13. Let $\vec{r}=(\vec{a}\times\vec{b})\sin x+(\vec{b}\times\vec{c})\cos y+2(\vec{c}\times\vec{a})$ where $\vec{a},\vec{b},\vec{c}$ are three noncoplanar vectors. If $\vec{r}$ is perpendicular to $\vec{a}+\vec{b}+\vec{c}$, then minimum value of x^2+y^2 is

(a) π^2 (b) $\pi^2/4$

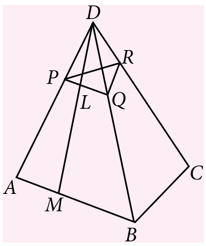
(c) $5\pi^2/4$ (d) none of these

14. In a quadrilateral $ABCD$, $\overrightarrow{AC}$ is the bisector of the $(\overrightarrow{AB}\cdot\overrightarrow{AD})$ which is $2\pi/3$, $15|\overrightarrow{AC}|=3|\overrightarrow{AB}|=5|\overrightarrow{AD}|$, then $\cos(\overrightarrow{BA}\cdot\overrightarrow{DC})$ is

(a) $-\dfrac{\sqrt{14}}{7\sqrt{2}}$ (b) $-\dfrac{\sqrt{21}}{7\sqrt{3}}$ (c) $\dfrac{2}{\sqrt{7}}$ (d) $\dfrac{2\sqrt{7}}{14}$

SOLUTIONS

1. (d) : Let $\dfrac{DP}{DA}=x,\ \dfrac{DQ}{DB}=y,\ \dfrac{DR}{DC}=z$



Let M be the midpoint, then

$$\frac{DL}{DM}=\frac{1}{3}$$

So, $\dfrac{ar(\Delta DLP)}{ar(\Delta DAM)}=\dfrac{DP\cdot DL}{DA\cdot DM}=\dfrac{x}{3}$,

$\dfrac{ar(\Delta DLQ)}{ar(\Delta DMB)}=\dfrac{DL\cdot DQ}{DM\cdot DB}=\dfrac{y}{3}$.

Since, $ar(\Delta DAM)=ar(\Delta DMB)=\dfrac{1}{2}ar(\Delta DAB)$

So, $2xy=2\cdot\dfrac{DP\cdot DQ}{DA\cdot DB}=\dfrac{ar(\Delta DPQ)}{\frac{1}{2}ar(\Delta DAB)}$

$=\dfrac{ar(\Delta DPL)}{ar(\Delta DAM)}+\dfrac{ar(\Delta DLQ)}{ar(\Delta DMB)}=\dfrac{x+y}{3}$

So, $\dfrac{1}{x}+\dfrac{1}{y}=6$

Similarly, $\dfrac{1}{y}+\dfrac{1}{z}=8$ and $\dfrac{1}{z}+\dfrac{1}{x}=10$

Solving, $x=\dfrac{1}{4},\ y=\dfrac{1}{2},\ z=\dfrac{1}{6}$

So, $\dfrac{V_{DPQR}}{V_{DABC}}=\dfrac{DP\cdot DQ\cdot DR}{DA\cdot DB\cdot DC}=xyz=\dfrac{1}{48}$ and

so the required ratio is 1 : 47.

2. (d) : Let S be the tetrahedron's circumsphere with radius R and centre O. Let P be the plane determined by points A, B, O. The intersection between P and S is a circle of radius R.

Now, using power of a point, to the point E, lying inside the circle C, we have $AE\cdot BE=R^2-OE^2$. Similarly, $BF\cdot CF=R^2-OF^2$, $CG\cdot AG=R^2-OG^2$. Hence, by given condition in question,

$$OE=OF=OG=OH=OK=OL$$

i.e. E, F, G, H, K, L lie on a sphere centred at O.

3. (d) : $(\vec{a}\times\vec{b})-(\vec{b}\times\vec{c})=\vec{0}\Rightarrow\vec{b}\times(\vec{a}+\vec{c})=\vec{0}$

So, $\vec{b}=\lambda(\vec{a}+\vec{c})$

Similarly, $\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$

$\Rightarrow\vec{c}\times(\vec{a}+\lambda(\vec{a}+\vec{c}))=\vec{0}\Rightarrow(1+\lambda)(\vec{c}\times\vec{a})=\vec{0}$

$\Rightarrow\lambda=-1$ so, $\vec{b}=-(\vec{a}+\vec{c})$ *i.e.*, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

4. (a) : Clearly $\vec{u}+\vec{v}+\vec{w}=\vec{0}$.

Hence, $\vec{u},\vec{v},\vec{w}$ form a triangle.

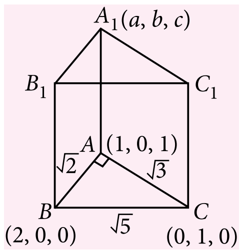
Now, consider $\vec{u}\cdot\vec{c}=(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{c})-(\vec{c}\cdot\vec{a})(\vec{b}\cdot\vec{c})=0$

i.e., $\vec{u}$ and $\vec{c}$ are orthogonal. Similarly, $\vec{v}$ is orthogonal to $\vec{a}$ and $\vec{w}$ to $\vec{b}$. Hence the sides of the triangle formed with $\vec{u},\vec{v},\vec{w}$ are perpendicular to the sides of the triangle formed with $\vec{a},\vec{b},\vec{c}$. This shows that the two triangles have equal angles and hence are similar.

5. (b) : Volume = Area of base × height

$\Rightarrow\ 3=\dfrac{1}{2}\times\sqrt{2}\times\sqrt{3}\times h$

$\Rightarrow\ h=\sqrt{6}$



$(A_1A)^2=h^2=6$

$\overline{A_1A}\cdot\overline{AB}=0$

$\overline{A_1A}\cdot\overline{AC}=0$

$\overline{AA_1}\cdot\overline{BC}=0$

Solving we get position vector of A_1 are (0, –2, 0) or (2, 2, 2)

6. (c) : Since $|\vec{a}+\vec{b}+\vec{c}|=1$

$\Rightarrow\ (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=1\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-1$

$\Rightarrow\ \cos\theta_1+\cos\theta_2+\cos\theta_3=-1$

So, at least one of $\cos\theta_1$, $\cos\theta_2$ and $\cos\theta_3$ must be negative.

7. (b) : $[ABC]=\sqrt{[OAB]^2+[OBC]^2+[OCA]^2}$,

where $[ABC]$ = area of triangle ABC

8. (c) : Let L_1, L_2, L_3 be the mutually perpendicular lines and $P(x_0, y_0, z_0)$ be their point of concurrence.

If L_1 cuts the x-axis at $A(a, 0, 0)$, L_2 meets the y-axis at $B(0, b, 0)$ and $C(0, 0, c) \in L_3$, then $L_1 \parallel (x_0 - a, y_0, z_0)$, $L_2 \parallel (x_0, y_0 - b, z_0)$ and $L_3 \parallel (x_0, y_0, z_0 - c)$.
Hence

$$\left.\begin{array}{l} x_0(x_0 - a) + y_0(y_0 - b) + z_0^2 = 0 \\ x_0^2 + (y_0 - b)y_0 + z_0(z_0 - c) = 0 \end{array}\right\}$$

$$x_0(x_0 - a) + y_0^2 + z_0(z_0 - c) = 0$$

Eliminating a and b from the above equations, we get

$$x_0^2 + y_0^2 + z_0^2 - 2cz_0 = 0$$

9. (c) : $\overrightarrow{AB} \times \overrightarrow{AC} = -\hat{i} + 2\hat{j} + \hat{k}$

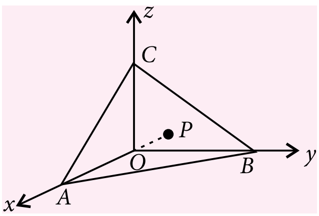
Height $= \dfrac{\left|\overrightarrow{AD}.(\overrightarrow{AB} \times \overrightarrow{AC})\right|}{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|} = \dfrac{2\sqrt{2}}{\sqrt{3}}$

10. (d) : Here $OP = \sqrt{h^2 + k^2 + l^2} = p$

$\therefore$ DRs of OP are

$$\frac{h}{\sqrt{h^2+k^2+l^2}}, \frac{k}{\sqrt{h^2+k^2+l^2}}, \frac{l}{\sqrt{h^2+k^2+l^2}} \text{ or } \frac{h}{p}, \frac{k}{p}, \frac{l}{p}$$

Since OP is normal to the plane, therefore equation of plane is



$$\frac{h}{p}x + \frac{k}{p}y + \frac{l}{p}z = p$$

or $\quad hx + ky + lz = p^2$

$$\therefore \quad A\left(\frac{p^2}{h}, 0, 0\right), B\left(0, \frac{p^2}{k}, 0\right), C\left(0, 0, \frac{p^2}{l}\right)$$

Now, Area of $\Delta ABC, \Delta = \sqrt{A_{xy}^2 + A_{yz}^2 + A_{zx}^2}$,

[Where, A_{xy}^2 is area of projection of ΔABC on xy plane = area of ΔAOB]

$$\text{Now, } A_{xy} = \frac{1}{2}\begin{vmatrix} p^2/h & 0 & 1 \\ 0 & p^2/k & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{p^4}{2|hk|}$$

Similarly, $A_{yz} = \dfrac{p^4}{2|kl|}$ and $A_{zx} = \dfrac{p^4}{2|lh|}$

$$\therefore \quad \Delta^2 = A_{xy}^2 + A_{yz}^2 + A_{zx}^2 \Rightarrow \Delta = \frac{p^5}{2hkl}$$

11. (a) : Let θ be the required angle then θ will be the angle between $\vec{a}$ and $\vec{b} + \vec{c}$ ($\vec{b} + \vec{c}$ lies along the angular bisector of $\vec{a}$ and $\vec{b}$)

$$\cos\theta = \frac{\vec{a}\cdot(\vec{b}+\vec{c})}{|\vec{a}||\vec{b}+\vec{c}|} = \frac{2\cos\alpha}{\sqrt{2+2\cos\alpha}} = \frac{\cos\alpha}{\cos\frac{\alpha}{2}}$$

$$\theta = \cos^{-1}\left(\frac{\cos\alpha}{\cos\alpha/2}\right)$$

12. (c) : $\vec{c}\cdot\vec{a} = ((\vec{a}\times\vec{c}) + \vec{b})\cdot\vec{a} = \vec{b}\cdot\vec{a}$

$\vec{b}\times\vec{c} = (\vec{b}\cdot\vec{c})\cdot\vec{a} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

$\therefore \; [\vec{a}\;\vec{b}\;\vec{c}] = \vec{b}\cdot\vec{c} - (\vec{a}\cdot\vec{b})\cdot(\vec{a}\cdot\vec{c})$

Also $\vec{c}\cdot\vec{b} = 1 - [\vec{a}\;\vec{b}\;\vec{c}]$

$\therefore \; 2[\vec{a}\;\vec{b}\;\vec{c}] = 1 - (\vec{a}\cdot\vec{b})^2 \le 1 \quad \therefore \quad [\vec{a}\;\vec{b}\;\vec{c}] \le \dfrac{1}{2}$

13. (c) : $\vec{r} = (\vec{a}\times\vec{b})\sin x + (\vec{b}\times\vec{c})\cos y + 2(\vec{c}\times\vec{a})$

Also, $\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$

$\Rightarrow \; [\vec{a}\;\vec{b}\;\vec{c}](\sin x + \cos y + 2) = 0$

$\because \quad [\vec{a}\;\vec{b}\;\vec{c}] \ne 0 \;\Rightarrow\; \sin x + \cos y = -2$

This is possible only when $\sin x = -1$ and $\cos y = -1$

For $x^2 + y^2$ to be minimum $x = -\pi/2$ and $y = \pi$

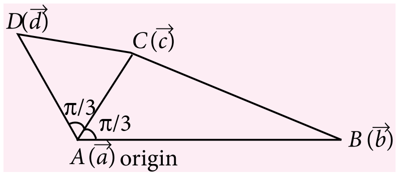
$\therefore \quad$ Minimum value of $(x^2 + y^2)$ is $\dfrac{\pi^2}{4} + \pi^2 = \dfrac{5\pi^2}{4}$

14. (c) : Given $15|\overrightarrow{AC}| = 3|\overrightarrow{AB}| = 5|\overrightarrow{AD}|$

Let $|\overrightarrow{AC}| = \lambda > 0$

$\therefore \; |\overrightarrow{AB}| = 5\lambda$

and $|\overrightarrow{AD}| = 3\lambda$



Now, $\cos(\overrightarrow{BA}\cdot\overrightarrow{CD}) = \dfrac{\overrightarrow{BA}\cdot\overrightarrow{CD}}{|\overrightarrow{BA}||\overrightarrow{CD}|} = \dfrac{\vec{b}\cdot(\vec{c}-\vec{d})}{|\vec{b}||\vec{c}-\vec{d}|}$ ...(i)

Now, numerator of (i) $= \vec{b}\cdot\vec{c} - \vec{b}\cdot\vec{d}$

$$= |\vec{b}||\vec{c}|\cos\frac{\pi}{3} - |\vec{b}||\vec{d}|\cos\frac{2\pi}{3}$$

$$= (5\lambda)(\lambda)\frac{1}{2} + (5\lambda)(3\lambda)\frac{1}{2} = \frac{5\lambda^2 + 15\lambda^2}{2} = 10\lambda^2$$

Denominator of (i) $= |\vec{b}||\vec{c}-\vec{d}|$

Now $|\vec{c}-\vec{d}|^2 = \vec{d}^2 + \vec{c}^2 - 2\vec{c}\cdot\vec{d}$

$= 9\lambda^2 + \lambda^2 - 2(\lambda)(3\lambda) \times 1/2 = 10\lambda^2 - 3\lambda^2 = 7\lambda^2$

$\therefore \quad |\vec{d}-\vec{c}| = \sqrt{7}\,\lambda$

Denominator of (i) $= (5\lambda)(\sqrt{7}\lambda) = 5\sqrt{7}\lambda^2$

$$\therefore \quad \cos(\overrightarrow{BA}\cdot\overrightarrow{CD}) = \frac{10\lambda^2}{5\sqrt{7}\lambda^2} = \frac{2}{\sqrt{7}}$$

MPP-6 CLASS XI ANSWER KEY

1.	(b)	2.	(a)	3.	(d)	4.	(b)	5.	(a)
6.	(b)	7.	(a,b,c,d)			8.	(a,b)	9.	(a,b,c)
10.	(a,b,d)			11.	(a,c)	12.	(a,b,c)	13.	(a,d)
14.	(b)	15.	(c)	16.	(c)	17.	$(\sqrt{7/8})$		
18.	(5)	19.	(8)	20.	(1)				

GEAR UP FOR JEE MAIN 2019

JEE Main I between 6th to 20th January and JEE Main II between 6th to 20th April 2019

KEY POINTS

Equation of circle

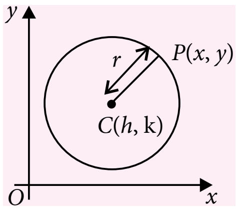
Let $C(h, k)$ be the centre of the circle and $CP(= r)$ be the radius of circle, then equation of circle is

$$(x - h)^2 + (y - k)^2 = r^2 \quad ...(i)$$

Now, if origin (0, 0) be the centre of circle, then eq. (i) becomes,

$$x^2 + y^2 = r^2 \quad ...(ii)$$

The area of the circle is given by πr^2 sq.unit.



General Equation of Circle

The general equation of second degree may represents a circle, if the coefficient of x^2 and coefficient of y^2 are identical and the coefficient of xy becomes zero. *i.e.*,

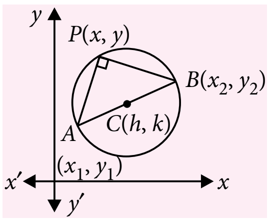
$$ax^2 + by^2 + 2hxy + + 2gx + 2fy + c = 0 \quad ...(i)$$

represents a circle, if (*a*) $a = b$ *i.e.*, coefficient of x^2 = coefficient of y^2 and (*b*) $h = 0$ *i.e.*, coefficient of $xy = 0$, then Eq.(i) reduces as, $x^2 + y^2 + 2gx + 2fy + c = 0$ whose centre and radius are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$ respectively.

- **Equation of circle in diameter form**

  Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of the given circle and let $P(x, y)$ be any point on the circle.

  $\therefore$ From figure, $\angle APB = 90°$

  $\therefore$ Slope of AP,

  $$m_1 = \left(\frac{y - y_1}{x - x_1}\right) \text{ and slope of } BP, m_2 = \left(\frac{y - y_2}{x - x_2}\right)$$

  For perpendicular, $m_1 \cdot m_2 = -1$

  $\therefore$ $AP \cdot BP = -1$

  $$\Rightarrow \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

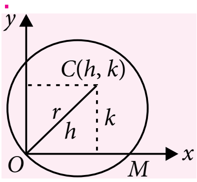
  $\Rightarrow$ $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$,

  which is the required equation of circle in diameter form.

Equation of circle in different cases :

- **Case I :** When the circle passes through the origin (0, 0) : Let the equation of circle be

  $(x - h)^2 + (y - k)^2 = r^2 \quad ....(i)$

  $\because$ It passes through origin (0, 0)

  $\therefore$ $h^2 + k^2 = r^2$

  $\therefore$ Equation (i) becomes,

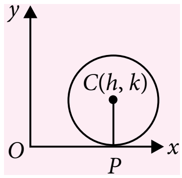
  $(x - h)^2 + (y - k)^2 = h^2 + k^2$

  $\Rightarrow$ $x^2 + h^2 - 2hx + y^2 - 2ky + k^2 = h^2 + k^2$

  $\Rightarrow$ $x^2 + y^2 - 2hx - 2ky = 0$

- **Case II :** When the circle touches x-axis:

  Let the centre of circle be $C(h, k)$, and it touches x- axis at point P, then the radius of circle is $CP = |k|$

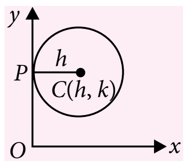
  $\therefore$ Equation of circle is

  $(x - h)^2 + (y - k)^2 = (CP)^2 = k^2$

  or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$

- **Case III :** When the circle touches y-axis:

  Let the centre of circle be $C(h, k)$ and it touches y-axis at point P, then the radius $CP = |h|$

  $\therefore$ Equation of circle is

  $(x - h)^2 + (y - k)^2 = (CP)^2 = h^2$

  or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$

- **Case IV :** When the circle touches both axis:

  In this case $|h| = |k| = \alpha$.

  Then the equation of circle is

  $(x - h)^2 + (y - k)^2 = r^2$ where, $|h| = |k| = |r| = \alpha$

  $\therefore$ $(x \pm \alpha)^2 + (y \pm \alpha)^2 = \alpha^2$

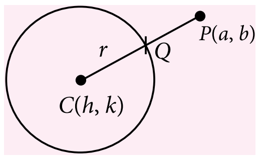
  or $x^2 + y^2 \pm 2\alpha x \pm 2\alpha y + \alpha^2 = 0$

Position of a point with respect to a circle

Let $C(h, k)$ be the centre and r be the radius of the circle and $P(a, b)$ be any point in the plane of the circle, then three cases arises *i.e.*,

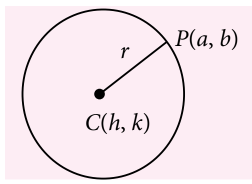
- **Case I :** Let 'P' lies outside the circle, then equation of circle is

  $(a - h)^2 + (b - k)^2 > r^2$

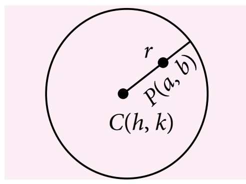
- **Case II :** Let point 'P' lies on the circle, then equation of circle is

  $(a - h)^2 + (b - k)^2 = r^2$

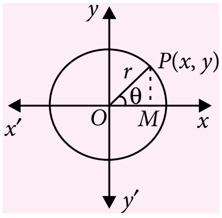
- **Case III :** Let point 'P' lies inside the circle, then equation of circle is

  $(a - h)^2 + (b - k)^2 < r^2$

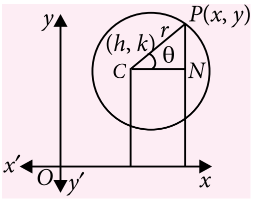
Equation of circle in parametric form

- **Case I :** Let $P(x, y)$ be any point on the circle $x^2 + y^2 = r^2$, then from fig. $\angle MOP = \theta$. On resolving the components, we get



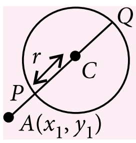
$x = OM = r\cos\theta$ ....(i)
and $y = PM = r\sin\theta$ ....(ii)
Here eqs. (i) and (ii) are the required parametric form of the circle $x^2 + y^2 = r^2$, where 'θ' is a parameter.

Case II : Parametric form of equation of circle, if (h, k) is the centre and r being the radius is



$x = h + r\cos\theta$,
$y = k + r\sin\theta$, $0 \le \theta \le 2\pi$
where θ being the parameter.

- **The least and greatest distance of a point from a circle**
Let $S = 0$ be a circle and $A(x_1, y_1)$ be a point. If the diameter of the circle is passing through the circle at P and Q, then $AP = AC - r$ = least distance. $AQ = AC + r$ = greatest distance where r is the radius and C is the centre of circle.

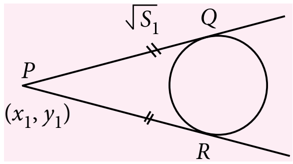


Condition of tangency

- A line $L = 0$ touches the circle $S = 0$, if length of perpendicular drawn from the centre of the circle to the line is equal to radius of the circle *i.e.*, $p = r$. This is the condition of tangency for the line $L = 0$.
Circle $x^2 + y^2 = a^2$ will touch the line $y = mx + c$ if $c = \pm a\sqrt{1+m^2}$
(a) If $a^2(1 + m^2) - c^2 > 0$ line will meet the circle at real and different points.
(b) If $c^2 = a^2(1 + m^2)$ line will touch the circle.
(c) If $a^2(1 + m^2) - c^2 < 0$ line will meet circle at two imaginary points (*i.e.* will never meet the circle).

Equation of tangent and normal

- **Equation of tangent :** The equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at a point (x_1, y_1) is
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
or $T = 0$
The equation of tangent to circle $x^2 + y^2 = a^2$ at point (x_1, y_1) is $xx_1 + yy_1 = a^2$.
- **Slope form :** From condition of tangency for every value of m, the line $y = mx \pm a\sqrt{1+m^2}$ is a tangent of the circle $x^2 + y^2 = a^2$ and its point of contact is
$$\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$$
- **Equation of normal :** Normal to a curve at any point P of a curve is the straight line passing through P and is perpendicular to the tangent at P. The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$
- **Length of tangent :** From any point, say $P(x_1, y_1)$ two tangents can be drawn to a circle which are real, coincident or imaginary according as P lies outside, on or inside the circle.
Let PQ and PR be the two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then $PQ = PR$ is called the length of tangent drawn from point P and is given by
$$PQ = PR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$



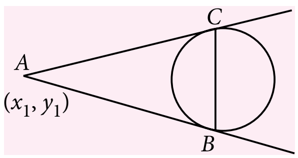
- **Pair of tangents :** From a given external point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle, $S = x^2 + y^2 + 2gx + 2fy + c = 0$.
Their combined equation is $SS_1 = T^2$, where $S = 0$ is the equation of circle, $T = 0$ is the equation of the tangent at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.

Director circle

The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle. Let the circle be $x^2 + y^2 = a^2$, then equation of pair of tangents to a circle from a point (x_1, y_1) is
$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$
If this represents a pair of perpendicular lines then coefficient of x^2 + coefficient of $y^2 = 0$
i.e., $(x_1^2 + y_1^2 - a^2 - x_1^2) + (x_1^2 + y_1^2 - a^2 - y_1^2) = 0$
$\Rightarrow \; x_1^2 + y_1^2 = 2a^2$
Hence the equation of director circle is $x^2 + y^2 = 2a^2$.
Obviously, director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.
Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is
$$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$$

Chord of contact

The chord joining the two points of contact of tangents to a circle drawn from any external point A is called chord of contact of A with respect to the given circle. Let the given point is $A(x_1, y_1)$ and the circle is $S = 0$ then equation of the chord of contact is



$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ... (i)
Note : (i) It is clear from the above that the equation of the chord of contact coincides with the equation of the tangent, if the point (x_1, y_1) lies on the circle.

(ii) The length of chord of contact $= 2\sqrt{r^2 - p^2}$

(iii) Area of $\Delta ABC = \dfrac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2}$

Equation of a chord whose middle point is given

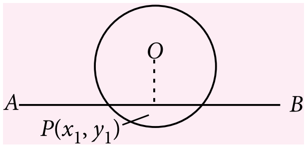
We have the circle $x^2 + y^2 = a^2$ and middle point of chord is $P(x_1, y_1)$.

Slope of the line $OP = \dfrac{y_1}{x_1}$; slope of $AB = -\dfrac{x_1}{y_1}$

So equation of chord is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

or $\quad xx_1 + yy_1 = x_1^2 + y_1^2$



which can be represented by $T = S_1$.

Common chord of two circles

The line joining the points of intersection of two circles is called the common chord. If the equation of two circles is

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1 y + c_1 = 0$$
$$S_2 = x^2 + y^2 + 2g_2x + 2f_2 y + c_2 = 0$$

then equation of common chord is

$S_1 - S_2 = 0 \Rightarrow 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

The length of the common chord is

$$2\sqrt{r_1^2 - p_1^2} = 2\sqrt{r_2^2 - p_2^2}$$

where p_1 and p_2 are the length of perpendicular drawn from the centre to the chord.

Angle of intersection of two circles

The angle of intersection between two circles $S = 0$ and $S' = 0$ is defined as the angle between their tangents at their point of intersection. If

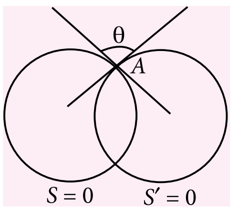
$$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

or $$\cos\theta = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$$

- **Condition of orthogonality :** If the angle of intersection of the two circles is 90° then such circles are called orthogonal circles and condition for orthogonal circles and condition for orthogonality is $2g_1g_2 + 2f_1 f_2 = c_1 + c_2$. When the two circles intersect orthogonally then the length of tangent on one circle from the centre of other circle is equal to the radius of the other circle.



Power of a point with respect to a circle

The power of a point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1 where

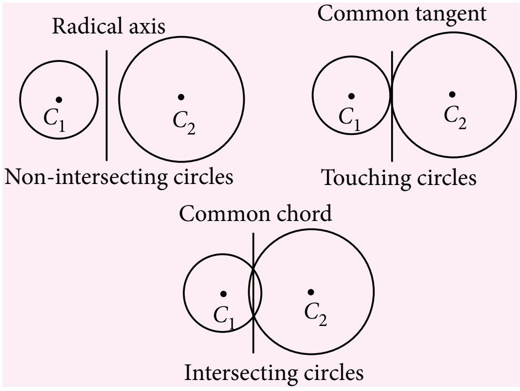
$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = 0$$

Radical axis

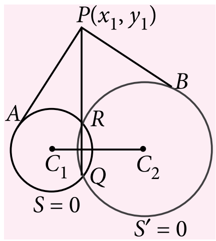
The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal in length.

Some properties of the radical axis are as follows :

- **The radical axis and common chord are identical :** Since the radical axis and common chord of the two circles $S = 0$ and $S' = 0$ are the same straight line $S - S' = 0$, they are identical. The only difference is that the common chord exists only if the circles intersect in two real points, while the radical axis exists for all pair of circles irrespective of their position.



The position of the radical axis of the two circles geometrically is shown below:



From Euclidean geometry, $(PA)^2 = PR \cdot PQ = (PB)^2$

- The radical axis is perpendicular to the straight line which joins the centres of the circles.

Consider, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ... (ii)

Since $C_1 \equiv (-g, -f)$ and $C_2 \equiv (-g_1, -f_1)$ are the centres of the circles (i) and (ii), then slope of

$$C_1C_2 = \frac{-f_1 + f}{-g_1 + g} = \frac{f - f_1}{g - g_1} = m_1 \text{ (say)}$$

Equation of the radical axis is

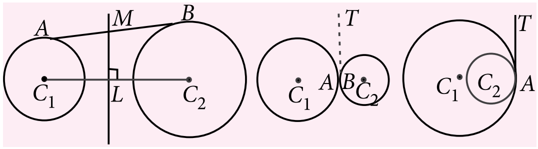
$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$

Slope of radical axis is $\dfrac{-(g - g_1)}{f - f_1} = m_2$ (say)

$\because \quad m_1m_2 = -1$

Hence C_1C_2 and radical axis are perpendicular to each other.

- **The radical axis bisects common tangents of two circles:** Let AB be the common tangent. If it meets the radical axis LM at M, then MA and MB are two tangents to the circles. Hence $MA = MB$ since lengths of tangents are equal from any point on radical axis. Hence radical axis bisects the common tangent AB.



If the two circles touch each other externally or internally, then A and B coincides. In this case the common tangent itself becomes the radical axis.

- **The radical axis of three circles taken in pairs are concurrent :** Let the equation of three circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{... (i)}$$
$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \text{... (ii)}$$
$$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad \text{... (iii)}$$

The radical axis of the above three circles taken in pairs are given by

$$S_1 - S_2 \equiv 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0 \quad \text{...(iv)}$$
$$S_2 - S_3 \equiv 2x(g_2 - g_3) + 2y(f_2 - f_3) + c_2 - c_3 = 0 \quad \text{... (v)}$$
$$S_3 - S_1 \equiv 2x(g_3 - g_1) + 2y(f_3 - f_1) + c_3 - c_1 = 0 \quad \text{... (vi)}$$

Adding (iv), (v) and (vi), we find LHS vanished identically. Thus the three lines are concurrent.

- If two circles cut the third circle orthogonally, then the radical axis of the two circles will pass through the centre of the third circle.

OR

The locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the two circles.

Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$ ...(iii)

Since (i) and (ii) both cut (iii) orthogonally

$\therefore \quad 2g_1g_3 + 2f_1f_3 = c_1 + c_3$

and $2g_2g_3 + 2f_2f_3 = c_2 + c_3$

Subtracting, we get

$2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$ ...(iv)

Now radical axis of (i) and (ii) is

$S_1 - S_2 = 0$ or $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

Since it will pass through the centre of (iii) circle

$\therefore \quad -2g_3(g_1 - g_2) - 2f_3(f_1 - f_2) + c_1 - c_2 = 0$

or $\quad 2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$ ...(v)

which is true by (iv).

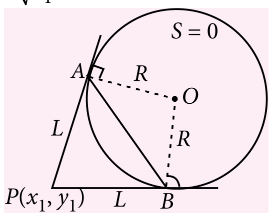
Some important results to remember

- If two conic sections $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$ will intersect each other in four concyclic points, if $\dfrac{a_1 - b_1}{a_2 - b_2} = \dfrac{h_1}{h_2}$.

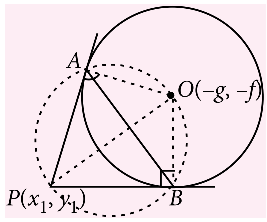
- The point of intersection of the tangents at the points $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ on the circle $x^2 + y^2 = a^2$ is

$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right)$$

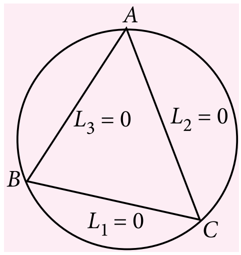
- Length of chord of contact is $AB = \dfrac{2LR}{\sqrt{(R^2+L^2)}}$ and area of the triangle formed by the pair of tangents and its chord of contact is $\dfrac{RL^2}{R^2+L^2}$ where R is the radius of the circle and L is the lengths of tangents from $P(x_1, y_1)$ on $S = 0$. Here $L = \sqrt{S_1}$.



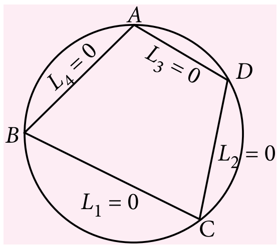
- Equation of the circle circumscribing the triangle PAB is $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$ where $O(-g, -f)$ is the centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$



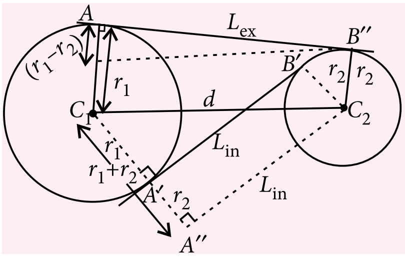
- Family of circles circumscribing a triangle whose sides are given by $L_1 = 0, L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.



- Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ is given by $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.



- Length of an external common tangent and internal common tangent to two circles is given by the length of external common tangent $L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$ and length of internal common tangent $L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$ [Applicable only when $d > (r_1 + r_2)$] where d is the distance between the centres of circles and r_1 and r_2 are the radii of two circles.



- The locus of the middle point of a chord of a circle subtending a right angle at a given point will be a circle.
- The length of a side of an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ is $a\sqrt{3}$.
- The distance between the chord of contact of tangents to $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is $\dfrac{|g^2+f^2-c|}{2\sqrt{(g^2+f^2)}}$
- The shortest chord of a circle passing through a point P inside the circle is the chord whose middle point is P.
- The length of transverse common tangent < the length of direct common tangent.
- The angle between the two tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\dfrac{a}{\sqrt{S_1}}\right)$; where $S_1 = x_1^2 + y_1^2 - a^2$.

PROBLEMS

1. The equation of the circle of radius 5 in the first quadrant which touches x-axis and the line $4y = 3x$ is
(a) $x^2 + y^2 - 24x - y - 25 = 0$
(b) $x^2 + y^2 - 30x - 10y + 225 = 0$
(c) $x^2 + y^2 - 16x - 18y + 64 = 0$
(d) $x^2 + y^2 - 20x - 12y + 144 = 0$

2. Find the equation of the circle which passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is $(2, -3)$.
(a) $(x-2)^2+(y+3)^2=(\sqrt{109})^2$
(b) $(x+2)^2-(y-3)^2=(\sqrt{109})^2$
(c) $(x-2)^2-(y+3)^2=(\sqrt{109})^2$
(d) $(x-2)^2-(y-3)^2=(\sqrt{109})^2$

3. If θ is the angle between the tangents from $(-1, 0)$ to the circle $x^2 + y^2 - 5x + 4y - 2 = 0$, then θ is equal to
(a) $2\tan^{-1}\left(\dfrac{7}{4}\right)$ (b) $\tan^{-1}\left(\dfrac{7}{4}\right)$
(c) $2\cot^{-1}\left(\dfrac{7}{4}\right)$ (d) $\cot^{-1}\left(\dfrac{7}{4}\right)$

4. The centre of a circle is $(2, -3)$ and the circumference is 10π. Then the equation of the circle is
(a) $x^2 + y^2 + 4x + 6y + 12 = 0$
(b) $x^2 + y^2 - 4x + 6y + 12 = 0$
(c) $x^2 + y^2 - 4x + 6y - 12 = 0$
(d) $x^2 + y^2 - 4x - 6y - 12 = 0$

5. The equation of the circle which passes through the intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ and whose centre lies on $13x + 30y = 0$ is
(a) $x^2 + y^2 + 30x - 13y - 25 = 0$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $2x^2 + 2y^2 + 30x - 13y - 25 = 0$
(d) $x^2 + y^2 + 30x - 13y + 25 = 0$

6. The equation of the circle on the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y + b)^2 = b^2$ as diameter, is
(a) $x^2 + y^2 = 2ab(bx + ay)$ (b) $x^2 + y^2 = bx + ay$
(c) $(a^2 + b^2)(x^2 + y^2) = 2ab(bx - ay)$
(d) $(a^2 + b^2)(x^2 + y^2) = 2(bx + ay)$

7. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for
(a) exactly one value of a (b) no value of a
(c) infinitely many values of a
(d) exactly two values of a

8. To which of the following circles, the line $y - x + 3 = 0$ is normal at the point $\left(3+\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$?
(a) $\left(x-3-\dfrac{3}{\sqrt{2}}\right)^2+\left(y-\dfrac{3}{\sqrt{2}}\right)^2=9$
(b) $\left(x-\dfrac{3}{\sqrt{2}}\right)^2+\left(y-\dfrac{3}{\sqrt{2}}\right)^2=9$
(c) $x^2 + (y - 3)^2 = 9$ (d) $(x - 3)^2 + y^2 = 9$

9. The equation of the circle which touches both the axes in I quadrant and whose radius is a, is
(a) $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
(b) $x^2 + y^2 + ax + ay - a^2 = 0$
(c) $x^2 + y^2 + 2ax + 2ay - a^2 = 0$
(d) $x^2 + y^2 - ax - ay + a^2 = 0$

10. The equation of pair of tangents drawn from the point $(0, 1)$ to the circle $x^2 + y^2 - 2x + 4y = 0$ is
(a) $4x^2 - 4y^2 + 6xy + 6x + 8y - 4 = 0$
(b) $4x^2 - 4y^2 + 6xy - 6x + 8y - 4 = 0$
(c) $x^2 - y^2 + 3xy - 3x + 2y - 1 = 0$
(d) $x^2 - y^2 + 6xy - 6x + 8y - 4 = 0$

11. The equation of the circle which passes through points of intersection of circles $x^2 + y^2 + 4x - 5y + 3 = 0$ and $x^2 + y^2 + 2x + 3y - 3 = 0$ and point $(-3, 2)$ is
(a) $x^2 + y^2 + 8x + 13y - 3 = 0$
(b) $4x^2 + 4y^2 + 13x - 8y + 3 = 0$
(c) $x^2 + y^2 - 13x - 8y + 3 = 0$
(d) $x^2 + y^2 - 13x + 8y + 3 = 0$

Solution Sender of Maths Musing

SET-189
- N. Jayanthi (Hyderabad)

SET-188
- Devjit Acharjee (West Bengal)

12. Tangents are drawn to the circle $x^2 + y^2 = 9$ at the points where it is met by the circle $x^2 + y^2 + 3x + 4y + 2 = 0$. The point of intersection of these tangents will be

(a) $\left(\frac{8}{11}, \frac{13}{11}\right)$ (b) $\left(-\frac{27}{11}, -\frac{36}{11}\right)$

(c) $\left(\frac{11}{27}, \frac{11}{36}\right)$ (d) $\left(\frac{3}{4}, 1\right)$

13. Suppose that two circles C_1 and C_2 in a plane have no points in common. Then

(a) there are exactly two line tangent to both C_1 and C_2

(b) there are exactly 3 lines tangent to both C_1 and C_2

(c) there are no lines tangent to both C_1 and C_2 or there are exactly two lines tangent to both C_1 and C_2

(d) there are no lines tangent to both C_1 and C_2 or there are exactly four lines tangent to both C_1 and C_2

14. The equation of the circle which passes through the origin and cuts orthogonally each of the two circles $x^2 + y^2 - 6x + 8 = 0$ and $x^2 + y^2 - 2x - 2y - 7 = 0$ is

(a) $3x^2 + 3y^2 - 8x - 13y = 0$

(b) $3x^2 + 3y^2 - 8x + 29y = 0$

(c) $3x^2 + 3y^2 + 8x + 29y = 0$

(d) $3x^2 + 3y^2 - 8x - 29y = 0$

15. For the two circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$ there is/are

(a) one pair of common tangents

(b) only one common tangent

(c) three common tangents

(d) no common tangent

16. Find the equation of the circle passing through the point (2, 1) and touching the line $x + 2y - 1 = 0$ at the point (3, –1).

(a) $3(x^2 + y^2) - 23x - 4y + 35 = 0$

(b) $x^2 - y^2 + 23x + 4y - 35 = 0$

(c) $2x^2 - 2y^2 - 23x - 4y + 35 = 0$

(d) None of these

17. If equation $x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle, then the condition for that circle to pass, through three quadrants only but not passing through the origin is

(a) $f^2 > c, g^2 > c, c > 0$

(b) $g^2 > c, f^2 < c, c > 0, h = 0$

(c) $f^2 > c, g^2 > c, c > 0, h = 0$

(d) $g^2 < c, f^2 < c, c < 0, h = 0$

18. The equation of the circle which has a tangent $2x - y - 1 = 0$ at (3, 5) on it and with the centre on $x + y = 5$, is

(a) $x^2 + y^2 + 6x - 16y + 28 = 0$

(b) $x^2 + y^2 - 6x - 16y - 28 = 0$

(c) $x^2 + y^2 + 6x + 6y - 28 = 0$

(d) $x^2 + y^2 - 6x - 6y - 28 = 0$

19. The distance from the centre of the circle $x^2 + y^2 = 2x$ to straight line passing through the points of intersection of the two circles $x^2 + y^2 + 5x - 8y + 1 = 0$ and $x^2 + y^2 - 3x + 7y - 25 = 0$ is

(a) 1/3 (b) 2 (c) 3 (d) 1

20. Two circles with radii r_1 and $r_2 (r_1 > r_2 \geq 2)$ touch each other externally. If θ be the angle between the direct common tangents, then

(a) $\theta = \sin^{-1}\left(\frac{r_1 + r_2}{r_1 - r_2}\right)$ (b) $\theta = 2\sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$

(c) $\theta = \sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$ (d) None of these

21. If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points then the value of a is

(a) 4 (b) –4 (c) 6 (d) –6

22. Two perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P. Then, the locus of P has the equation

(a) $x^2 + y^2 = 2a^2$ (b) $x^2 + y^2 = 3a^2$

(c) $x^2 + y^2 = 4a^2$ (d) None of these

23. The area of the triangle formed by the tangents from an external point (h, k) to the circle $x^2 + y^2 = a^2$ and the chord of contact, is

(a) $\frac{1}{2}a\left(\frac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}}\right)$ (b) $\frac{a(h^2 + k^2 - a^2)^{3/2}}{2(h^2 + k^2)}$

(c) $\frac{a(h^2 + k^2 - a^2)^{3/2}}{(h^2 + k^2)}$ (d) None of these

24. The locus of a point which moves so that the ratio of the length of the tangents to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is 2 : 3, is

(a) $5x^2 + 5y^2 + 60x - 7 = 0$ (b) $5x^2 + 5y^2 - 60x - 7 = 0$

(c) $5x^2 + 5y^2 + 60x + 7 = 0$ (d) $5x^2 + 5y^2 + 60x + 12 = 0$

25. The circle S_1 with centre $C_1(a_1, b_1)$ and radius r_1 touches externally the circle S_2 with centre $C_2(a_2, b_2)$ and radius r_2. If the tangent at their common point passes through the origin, then

(a) $(a_1^2 + a_2^2) + (b_1^2 + b_2^2) = r_1^2 + r_2^2$

(b) $(a_2^2 - a_1^2) + (b_2^2 - b_1^2) = r_2^2 - r_1^2$

(c) $(a_1^2 - b_2^2) + (a_2^2 + b_2^2) = r_1^2 + r_2^2$

(d) $(a_1^2 - b_1^2) + (a_2^2 + b_2^2) = r_{12} + r_2^2$

26. If two circles, each of radius 5 unit, touch each other at (1, 2) and the equation of their common tangent is $4x + 3y = 10$, then equation of the circle a portion of which lies in all the quadrants, is

(a) $x^2 + y^2 - 10x - 10y + 25 = 0$

(b) $x^2 + y^2 + 6x + 2y - 15 = 0$

(c) $x^2 + y^2 + 2x + 6y - 15 = 0$

(d) $x^2 + y^2 + 10x + 10y + 25 = 0$

27. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is

(a) $8\sqrt{3}$ sq. units (b) $4\sqrt{3}$ sq. units

(c) $16\sqrt{3}$ sq. units (d) None of these

28. The equations of three circles are given:
$x^2 + y^2 = 1$, $x^2 + y^2 - 8x + 15 = 0$, $x^2 + y^2 + 10y + 24 = 0$. The coordinates of the point such that the tangents drawn from it to three circles are equal in length, are

(a) $\left(2, \frac{5}{2}\right)$ (b) $\left(2, \frac{-5}{2}\right)$

(c) $\left(-2, \frac{5}{2}\right)$ (d) $\left(-2, \frac{-5}{2}\right)$

29. If the tangents are drawn from any point on the line $x + y = 3$ to the circle $x^2 + y^2 = 9$, then the chord of contact passes through the point

(a) (3, 5) (b) (3, 3)

(c) (5, 3) (d) None of these

30. The slope of the tangent at the point (h, h) on the circle $x^2 + y^2 = a^2$ is

(a) 0 (b) 1

(c) –1 (d) dependent of h

SOLUTIONS

1. (b) : Let the centre of circle be $(g, 5)$.

$\therefore \frac{3(g)-4(5)}{\sqrt{3^2+4^2}} = 5 \Rightarrow 3g = 25 + 20 \Rightarrow g = 15$

$\therefore$ Equation of circle whose centre is (15, 5) and radius 5 is $(x - 15)^2 + (y - 5)^2 = 5^2$

$\Rightarrow x^2 - 30x + y^2 - 10y + 225 = 0$

2. (a) : Let P be the point of intersection of the lines AB and LM whose equations are respectively

$3x - 2y - 1 = 0$ ...(i) and $4x + y - 27 = 0$ .... (ii)

Solving (i) and (ii), we get $x = 5$, $y = 7$. So, coordinates of P are (5, 7). It is given that $C(2, -3)$ be the centre of the circle. Since the circle passes through P, therefore

$$CP = \text{radius} = \sqrt{(5-2)^2+(7+3)^2} \Rightarrow \text{radius} = \sqrt{109}$$

Hence the equation of the required circle is

$$(x-2)^2+(y+3)^2 = (\sqrt{109})^2$$

3. (a) : We know that, the angle between the two tangents from (α, β) to the circle $x^2 + y^2 = r^2$ is

$$2\tan^{-1}\frac{r}{\sqrt{S_1}}$$

Let $S = x^2 + y^2 - 5x + 4y - 2$

Here, $r = \sqrt{\left(-\frac{5}{2}\right)^2 + (2)^2 + 2} = \frac{7}{2}$

At point (–1, 0), $S_1 = (-1)^2 + (0)^2 - 5(-1) + 4(0) - 2 = 4$

$\therefore$ Required angle, $\theta = 2\tan^{-1}\frac{7/2}{\sqrt{4}} = 2\tan^{-1}\left(\frac{7}{4}\right)$

4. (c) : It is given, centre is (2, –3) and circumference of circle $= 10\pi \Rightarrow 2\pi r = 10\pi \Rightarrow r = 5$

$\therefore$ The equation of circle is $(x - 2)^2 + (y + 3)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 4x + 6y + 13 = 25$

$\Rightarrow x^2 + y^2 - 4x + 6y - 12 = 0$

5. (b) : Let the equation of circles be

$S_1 \equiv x^2 + y^2 + 13x - 3y = 0$ ... (i)

and $S_2 \equiv 2x^2 + 2y^2 + 4x - 7y - 25 = 0$ ... (ii)

The equation of intersecting circle is $\lambda S_1 + S_2 = 0$

$\Rightarrow \lambda(x^2+y^2+13x-3y)+\left(x^2+y^2+2x-\frac{7y}{2}-\frac{25}{2}\right)=0$ ...(iii)

$\therefore$ Centre $= \left(-\frac{(2+13\lambda)}{2(1+\lambda)}, \frac{(7/2)+3\lambda}{2(1+\lambda)}\right)$

$\because$ Centre lies on $13x + 30y = 0$.

$\therefore -13\left(\frac{2+13\lambda}{2}\right)+30\left(\frac{(7/2)+3\lambda}{2}\right)=0$

$\Rightarrow -26 - 169\lambda + 105 + 90\lambda = 0 \Rightarrow \lambda = 1$

Hence, putting the value of λ in (iii), we get required equation of circle as $4x^2 + 4y^2 + 30x - 13y - 25 = 0$

6. (c) : The equation of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y + b)^2 = b^2$ is

$I \equiv S_1 - S_2 = 0$

$\Rightarrow x^2 + a^2 - 2ax + y^2 - a^2 - x^2 - y^2 - b^2 - 2by + b^2 = 0$

$\Rightarrow ax + by = 0$ ... (i)

Now, the equation of required circle is $S_1 + \lambda L = 0$

$\therefore \{(x - a)^2 + y^2 - a^2\} + \lambda\{ax + by\} = 0$

$\Rightarrow x^2 + y^2 + x(a\lambda - 2a) + \lambda by = 0$ ... (ii)

Since, (i) is a diameter of (ii).

$\therefore a\left(-\frac{a\lambda-2a}{2}\right)+b\left(-\frac{\lambda b}{2}\right)=0 \Rightarrow \lambda = \frac{2a^2}{a^2+b^2}$

On putting the value of λ in (ii), we get

$(a^2 + b^2)(x^2 + y^2) = 2ab(bx - ay)$

which is the required equation of circle.

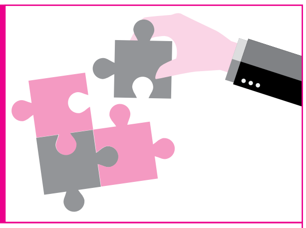
7. (b) : $S_1 - S_2 = 5ax + (c - d)y + a + 1 = 0$ and $5x + by - a = 0$ must represent the same line.

$\therefore \frac{a}{1} = \frac{c-d}{b} = \frac{a+1}{-a}$

$\Rightarrow ab = c - d$ and $a^2 + a + 1 = 0$

Thus, a is imaginary so no value of a exists.

PUZZLE CORNER



SOLUTION - SEPTEMBER 2018

5– 6	120× 5	4	2÷ 2	7+ 1	3
1	3+ 2	6	4	3	15+ 5
10+ 2	1	2÷ 3	6	7+ 5	4
3	4	1	25× 5	2	6
9+ 4	36× 3	5	1	9+ 6	2
5	6	2	7+ 3	4	1

8. (d) : Line must pass through the centre of the circle.

9. (a) : Required equation is $(x - a)^2 + (y - a)^2 = a^2$

$\Rightarrow \quad x^2 + y^2 - 2ax - 2ay + a^2 = 0$

10. (b) : Let $S = x^2 + y^2 - 2x + 4y$ then

$S_1 = 0^2 + 1^2 - 2 \cdot 0 + 4 \cdot 1 = 5$

$T = x \cdot 0 + y \cdot 1 - (x + 0) + 2(y + 1) = -x + 3y + 2$

The equation of the pair of tangents is $SS_1 = T^2$

$\Rightarrow \quad (x^2 + y^2 - 2x + 4y)5 = (-x + 3y + 2)^2$

$\Rightarrow \quad 4x^2 - 4y^2 + 6xy - 6x + 8y - 4 = 0$

11. (b) : The equation of circle through the points of intersection of given circles is

$x^2 + y^2 + 4x - 5y + 3 + \lambda(x^2 + y^2 + 2x + 3y - 3) = 0$

Since it passes through point $(-3, 2)$ also, therefore

$$-6 + 10\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{3}{5}$$

Hence equation of required circle is

$5x^2 + 5y^2 + 20x - 25y + 15 + 3x^2 + 3y^2 + 6x + 9y - 9 = 0$

$\Rightarrow \quad 8x^2 + 8y^2 + 26x - 16y + 6 = 0$

$\Rightarrow \quad 4x^2 + 4y^2 + 13x - 8y + 3 = 0$

12. (b) : Equation of common chord will be

$3x + 4y + 11 = 0 \qquad$ ...(i)

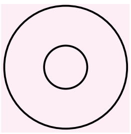
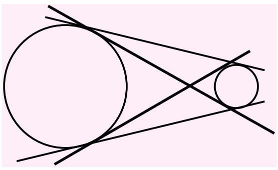
Let the point of intersection of the tangents be (α, β).

$\therefore$ Equation of the chord of contact of the tangents drawn from (α, β) to first circle will be

$x\alpha + y\beta = 9 \qquad$ ...(ii)

Since, (i) and (ii) are identical.

$$\therefore \quad \frac{3}{\alpha} = \frac{4}{\beta} = -\frac{11}{9} \quad \Rightarrow \quad (\alpha, \beta) = \left(\frac{-27}{11}, -\frac{36}{11}\right)$$

13. (d) :  or 

14. (b) : Let required equation of circle be

$x^2 + y^2 + 2gx + 2fy = 0$

Since, the above circle cuts the given circles orthogonally.

$\therefore \quad 2(-3g) + 2f(0) = 8 \;\Rightarrow\; 2g = -8/3$

and $-2g - 2f = -7 \;\Rightarrow\; 2f = 7 + \frac{8}{3} = \frac{29}{3}$

$\therefore$ Required equation of the circle is

$$x^2 + y^2 - \frac{8}{3}x + \frac{29}{3}y = 0 \;\Rightarrow\; 3x^2 + 3y^2 - 8x + 29y = 0$$

15. (d) : The centres and radii of given circles are

$C_1(0, 0),\; r_1 = 4,\; C_2(0, 1),\; r_2 = \sqrt{0 + 1} = 1$

Now, $C_1C_2 = \sqrt{0 + (0 - 1)^2} = 1$ and $r_1 - r_2 = 3$.

$\therefore \quad C_1C_2 < r_1 - r_2$

Hence, second circle lies inside the first circle, so no common tangent is possible.

16. (a) : Equation of circle is

$(x - 3)^2 + (y + 1)^2 + \lambda(x + 2y - 1) = 0$

Since, it passes through the point (2, 1).

$$\therefore \quad 1 + 4 + \lambda(2 + 2 - 1) = 0 \quad \Rightarrow \quad \lambda = -\frac{5}{3}$$

$$\therefore \quad \text{Circle is } (x-3)^2 + (y+1)^2 - \frac{5}{3}(x + 2y - 1) = 0$$

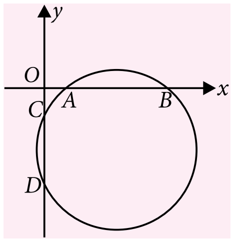
$\Rightarrow \quad 3x^2 + 3y^2 - 23x - 4y + 35 = 0$

17. (c) : Given circle is

$x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0 \qquad$ ...(i)

For (i) to represent a circle, $h = 0$

So, given circle is $x^2 + y^2 + 2gx + 2fy + c = 0 \qquad$ ...(ii)



For circle (ii) to pass through three quadrants only.

(I) $\quad AB > 0 \quad \Rightarrow \quad g^2 - c > 0$

(II) $\quad CD > 0 \quad \Rightarrow \quad f^2 - c > 0$

(III) Origin should be outside circle (ii).

$\therefore \quad c > 0$

From (I), (II) and (III), $g^2 > c, f^2 > c, c > 0$

$\therefore$ Required conditions are

$g^2 > c, f^2 > c, c > 0, h = 0$

18. (a) : Clearly, the centre of the circle lies on the line through the point (3, 5) perpendicular to the tangent $2x - y - 1 = 0$. The equation of such line is

$$(y - 5) = \frac{-1}{2}(x - 3) \;\Rightarrow\; x + 2y = 13 \qquad \text{...(i)}$$

Also, it is given that centre lies on the line

$x + y = 5 \qquad$ ...(ii)

Solving (i) and (ii), we obtain the coordinates of the centre of circle as $C \equiv (-3, 8)$

Also, radius of the circle $= \sqrt{36 + 9} = \sqrt{45}$

$\therefore$ Equation of the circle is

$(x + 3)^2 + (y - 8)^2 = (\sqrt{45})^2$

$\Rightarrow \quad x^2 + y^2 + 6x - 16y + 28 = 0$

19. (b) : The equation of the straight line passing through the points of intersection of given circles is

$(x^2 + y^2 + 5x - 8y + 1) - (x^2 + y^2 - 3x + 7y - 25) = 0$

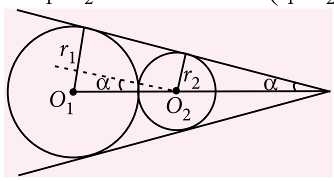
$\Rightarrow \quad 8x - 15y + 26 = 0 \qquad$ ...(i)

Also, centre of the circle $x^2 + y^2 - 2x = 0$ is (1, 0).

$\therefore$ Distance of the point (1, 0) from the straight line (i) is given by

$$d = \frac{|8(1) - 15(0) + 26|}{\sqrt{64 + 225}} = \frac{34}{17} = 2$$

20. (b) : $\sin\alpha = \dfrac{r_1 - r_2}{r_1 + r_2} \;\Rightarrow\; \theta = 2\sin^{-1}\left(\dfrac{r_1 - r_2}{r_1 + r_2}\right)$



21. (b) : Any second degree curve passing through the intersection of the given curves is

$$ax^2 + 4xy + 2y^2 + x + y + 5 + \lambda \times (ax^2 + 6xy + 5y^2 + 2x + 3y + 8) = 0$$

If it is a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$

$a(1 + \lambda) = 2 + 5\lambda$ and $4 + 6\lambda = 0$

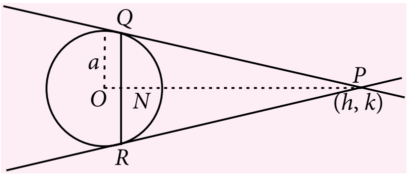
$$\Rightarrow \quad a = \frac{2+5\lambda}{1+\lambda} \text{ and } \lambda = -\frac{2}{3} \quad \Rightarrow \quad a = \frac{2-\frac{10}{3}}{1-\frac{2}{3}} = -4$$

22. (a) : We know that, if two perpendicular tangents to the circle $x^2 + y^2 = a^2$ meet at P, then the point P lies on a director circle. Thus, the equation of director circle to the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$

which is the required locus of point P.

23. (c) : Here, area of ΔPQR is required.

Now chord of contact with respect to circle $x^2 + y^2 = a^2$, and point (h, k) is $hx + ky - a^2 = 0$



Now, length of $\perp r$, $PN = \dfrac{h^2 + k^2 - a^2}{\sqrt{h^2 + k^2}}$

Also, $QR = 2\sqrt{a^2 - \dfrac{(a^2)^2}{h^2 + k^2}} = \dfrac{2a\sqrt{h^2 + k^2 - a^2}}{\sqrt{h^2 + k^2}}$

$$\therefore \text{ Area of } \Delta PQR = \frac{1}{2}(QR)(PN)$$

$$= \frac{1}{2} 2a \frac{\sqrt{h^2 + k^2 - a^2}}{\sqrt{h^2 + k^2}} \times \frac{(h^2 + k^2 - a^2)}{\sqrt{h^2 + k^2}}$$

$$= a \frac{(h^2 + k^2 - a^2)^{3/2}}{h^2 + k^2}$$

24. (c) : Let $P(x_1, y_1)$ be any point outside the circle. Length of tangent to the circle $x^2 + y^2 + 4x + 3 = 0$ is $\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}$ and length of tangent to the circle $x^2 + y^2 - 6x + 5 = 0$ is $\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}$

According to question, $\dfrac{\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}}{\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}} = \dfrac{2}{3}$

$$\Rightarrow \quad 9x_1^2 + 9y_1^2 + 36x_1 + 27 - 4x_1^2 - 4y_1^2 + 24x_1 - 20 = 0$$

$$\Rightarrow \quad 5x_1^2 + 5y_1^2 + 60x_1 + 7 = 0$$

$\therefore$ Locus of point P is $5x^2 + 5y^2 + 60x + 7 = 0$.

25. (b) : The two circles are

$$S_1 = (x - a_1)^2 + (y - b_1)^2 = r_1^2 \quad \text{... (i)}$$

$$S_2 = (x - a_2)^2 + (y - b_2)^2 = r_2^2 \quad \text{... (ii)}$$

The equation of the common tangent of these two circles is given by $S_1 - S_2 = 0$

$$\Rightarrow \quad 2x(a_1 - a_2) + 2y(b_1 - b_2) + (a_2^2 + b_2^2) - (a_1^2 + b_1^2) + r_1^2 - r_2^2 = 0$$

If this passes through the origin, then

$$(a_2^2 + b_2^2) - (a_1^2 + b_1^2) + r_1^2 - r_2^2 = 0$$

$$\Rightarrow \quad (a_2^2 - a_1^2) + (b_2^2 - b_1^2) = r_2^2 - r_1^2$$

26. (b) : The centres of the two circles will lie on the line through $P(1, 2)$ and perpendicular to the common tangent $4x + 3y = 10$. If C_1 and C_2 are the centres of these circles, then $PC_1 = 5 = r_1$ and $PC_2 = 5 = r_2$.

Also, C_1, C_2 lie on the line $\dfrac{x-1}{\cos\theta} = \dfrac{y-2}{\sin\theta} = r$, where $\tan\theta = \dfrac{3}{4}$.

When $r = r_1$, the coordinates of C_1 are

$$(5\cos\theta + 1, 5\sin\theta + 2) \text{ or } (5, 5) \text{ as } \cos\theta = \frac{4}{5}, \ \sin\theta = \frac{3}{5}$$

When $r = r_2$, the coordinates of C_2 are $(-3, -1)$.

The circle with centre $C_1(5, 5)$ and radius 5 touches both the coordinates axes and hence lies completely in the first quadrant.

SAMURAI SUDOKU

SOLUTION - SEPTEMBER 2018

9	8	5	6	1	3	4	7	2				7	4	9	1	8	5	2	6	3
7	3	2	4	8	5	9	6	1				1	2	5	4	6	3	7	9	8
4	6	1	9	2	7	8	5	3				8	3	6	9	7	2	5	4	1
1	7	8	5	3	6	2	9	4				5	8	7	2	1	6	9	3	4
2	4	6	7	9	1	5	3	8				9	1	2	7	3	4	6	8	5
3	5	9	8	4	2	6	1	7				4	6	3	8	5	9	1	2	7
5	1	3	2	6	4	7	8	9	2	6	1	3	5	4	6	2	7	8	1	9
6	9	4	1	7	8	3	2	5	8	4	7	6	9	1	5	4	8	3	7	2
8	2	7	3	5	9	1	4	6	5	3	9	2	7	8	3	9	1	4	5	6
						9	5	3	6	2	8	4	1	7						
						2	1	4	7	5	3	8	6	9						
						6	7	8	1	9	4	5	3	2						
8	1	3	7	6	5	4	9	2	3	1	6	7	8	5	1	6	3	2	9	4
2	6	9	4	8	1	5	3	7	9	8	2	1	4	6	2	7	9	8	5	3
5	4	7	3	2	9	8	6	1	4	7	5	9	2	3	5	4	8	7	6	1
9	5	6	2	4	7	1	8	3				5	9	2	7	1	4	6	3	8
3	7	4	1	9	8	6	2	5				3	6	8	9	2	5	1	4	7
1	2	8	6	5	3	7	4	9				4	7	1	8	3	6	9	2	5
7	8	2	9	1	6	3	5	4				6	1	9	4	5	7	3	8	2
4	3	5	8	7	2	9	1	6				8	5	7	3	9	2	4	1	6
6	9	1	5	3	4	2	7	8				2	3	4	6	8	1	5	7	9

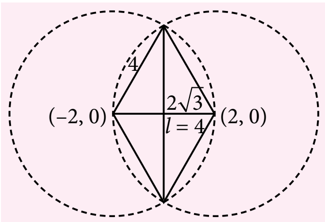
Solution Senders : *Ridhi Roy, Sandeepa Dhar (West Bengal)*

Therefore, the required circle has centre (–3, –1) and radius 5, so its equation is

$$(x + 3)^2 + (y + 1)^2 = 5^2 \Rightarrow x^2 + y^2 + 6x + 2y - 15 = 0$$

Since, the origin lies inside the circle, a portion of the circle lies in all the quadrants.

27. (a) : We have, circles with centre (2, 0) and (–2, 0) each with radius 4. So, y-axis is their common chord.



The inscribed rhombus has its diagonals equal to 4 and $4\sqrt{3}$.

$\therefore$ Area of rhombus $= \dfrac{d_1 d_2}{2} = 8\sqrt{3}$

28. (b) : Let (x_1, y_1) be the point. As the tangents from (x_1, y_1) to the first two circles are equal, (x_1, y_1) is on the radical axis of the circles, its equation being

$$S_1 - S_2 = (x^2 + y^2 - 1) - (x^2 + y^2 - 8x + 15) = 0$$

$\Rightarrow \quad 8x - 16 = 0 \quad \Rightarrow \quad x - 2 = 0 \qquad$ ...(i)

Similarly, (x_1, y_1) is on the radical axis of the second and third circle whose equation is

$S_2 - S_3 = x^2 + y^2 - 8x + 15 - (x^2 + y^2 + 10y + 24) = 0$

$\Rightarrow \quad 8x + 10y + 9 = 0 \qquad$ ...(ii)

Solving (i) and (ii), we get $x = 2$ and $y = -5/2$

$\therefore$ The required point is $\left(2, -\dfrac{5}{2}\right)$.

29. (b) : The coordinates of any point on the line $x + y = 3$ are $(k, 3 - k)$. The equation of chord of contact of tangents drawn from $(k, 3 - k)$ to the circle $x^2 + y^2 = 9$ is $kx + (3 - k) \cdot y = 9$

$\Rightarrow \quad (3y - 9) + k(x - y) = 0$

which clearly passes through the intersection of

$3y - 9 = 0$ and $x - y = 0$ *i.e.*, (3, 3).

30. (c) : The equation of the tangent at (h, h) to $x^2 + y^2 = a^2$ is $hx + hy = a^2$.

Therefore, slope of the tangent $= -h/h = -1$

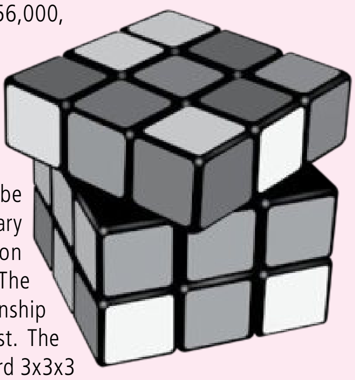
Cuber's Delight

Vako Marchelashvili, an 18-years-old student from Georgia, recently solved six Rubik's Cubes under water in one breath, setting a new Guinness World Record. Marchelashvili did it in just over 1:44 minutes. Here are some factoids about the 44-year-old invention that continues to be a favourite among the young and the old

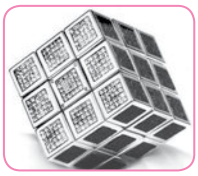
The cube was not invented as a toy. In 1974 Hungarian architect Erno Rubik wanted a working model to help explain three-dimensional geometry. He designed a "magic cube" made up of nine coloured squares on each side. It was heavier than the one used today. Rubik himself took a month to solve it

However, a robot is the real record holder: The LEGO Mindstorms-built cubestormer III machine (taking orders from [a] Galaxy S4 smart-phone) solved the puzzle in 3.25 seconds in 2014. The toy has even become a museum exhibit. New Jersey's Liberty Science Center hosted Beyond Rubik's Cube in November 2014.

The cube can be rearranged in more than 43 quintillion ways–43,252,003,274,489,856,000, according to some estimates.

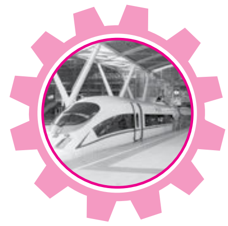


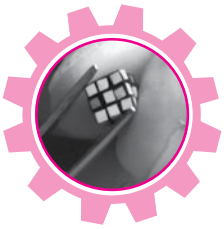
In 1979, toy specialist Tom Kremer saw the cube at fair in Nuremberg and convinced Ideal Toy Company to distribute it. The Rubik's Cube was released in 1980. By January 2009, more than 350 million units were sold worldwide. The first Rubik's World Championship was held in 1982 in Budapest. The fastest time to solve a standard 3x3x3 Rubik's Cube is 4.22 seconds by **Feliks Zemdegs** (Australia), according to the Guinness World Records.



The most expensive Rubik's Cube was created in 1995, with white diamonds, red rubies, green emeralds, purple amethysts and blue and yellow sapphires set in 18-karat gold. It is believed to be worth \$ 1.5m - but it is fully functional.

The largest Rubik's cube is 1.57m tall, 1.57m wide, 1.57m long and was made by Tony Fisher, in Ipswich, Suffolk, UK, as verified on 5 April 2016. He also the world's smallest– a mere 5.6mm wide.





There are records for solving the cube are many – depending on sizes and what you are doing simultaneously. There are records for solving the Rubik's Cube while running, cycling, juggling and while being blindfolded.

Courtesy : The Economic Times

CLASS XI Series 6

CBSE DRILL

Synopsis and Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2018-19.

Sequences and Series

SEQUENCES

A set of numbers arranged in a definite order according to a particular rule is called a sequence. Each number of the set is called a term of the sequence. A sequence is called finite or infinite according as number of terms in it is finite or infinite.

SERIES

Let $a_1, a_2, a_3, \ldots \ldots \ldots a_n$ be a given sequence. Then, the expression $a_1 + a_2 + a_3 + \ldots \ldots \ldots + a_n$ is called series associated with the given sequence. In other words, a series is the sum of the terms of a sequence.

ARITHMETIC PROGRESSION (A.P.)

- A sequence (finite or infinite) is called an arithmetic progression (A.P.) if the difference between any term and its preceeding term is constant. This constant is called common difference denoted by 'd' of the arithmetic progression.
- If the terms of a sequence are given by $a_1, a_2, a_3, \ldots, a_n$, then they will be in A.P. if $a_2 - a_1 = a_3 - a_2 = \ldots \ldots \ldots = a_n - a_{n-1} = d$, where d is the common difference.

General Term or n^{th} Term of an A.P.

- If a is the first term and d is the common difference of any A.P., then its general term is, $a_n = a + (n - 1)d$.
- If l is the last term of an A.P. consisting of n-terms with common difference d, then $l = a + (n - 1)d$.
- If l is the last term and d is the common difference, then general term from the end is, $a_n = l - (n - 1)d$.

Sum of n-terms of an A.P.

Sum of n terms is given by, $S_n = \frac{n}{2}[2a + (n - 1)d]$

or $\frac{n}{2}[a + l]$

where a, d and l denotes first term, common difference and last term of the A.P. respectively.

Note :

(i) Three numbers in A.P. can be taken as $a - d, a, a + d$.

(ii) Four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$.

(iii) Five numbers in A.P. can be taken as $a - 2d, a - d, a, a + d, a + 2d$.

Properties of an Arithmetic Progression

(i) If the same quantity be added to, or subtracted from, all the terms of an A.P., can the resulting progression is also an arithmetic progression.

(ii) If the corresponding terms of two arithmetic progressions be added or subtracted, the resulting progression is also an arithmetic progression.

(iii) If all the terms of an arithmetic progression be multiplied or divided by the same non-zero quantity, then the resulting progression is also an arithmetic progression.

Arithmetic Mean (A.M.)

- **A.M. between two numbers**
  Let a and b be two numbers, then arithmetic mean (A.M.) is given by $A = \frac{a+b}{2}$ such that a, A, b are in A.P.

- ***n* A.M.'s between two numbers**
  Let a and b be the two given numbers and $A_1, A_2, ..., A_n$ be n A.M.'s between them, then $d = \frac{b-a}{n+1}$

  $\therefore \quad A_1 = a + \frac{b-a}{n+1}, A_2 = a + \frac{2(b-a)}{n+1}, ...$

  $..., A_n = a + \frac{n(b-a)}{n+1}$

  $\therefore$ Sum of n A.M.'s $= A_1 + A_2 + ... + A_n = n\left(\frac{a+b}{2}\right)$

Geometric Progression (G.P.)

A sequence (finite or infinite) of non-zero numbers is called a geometric progression (G.P.) if the ratio of any term to its preceeding term is constant.
The non-zero constant is called common ratio denoted by 'r' of the geometric progression.
A G.P. having first term a and common ratio r can be written as $a, ar, ar^2, ..., ar^{n-1}$ or $a, ar, ar^2, ..., ar^{n-1}, ...$ to ∞ according as the G.P. is finite or infinite respectively.

General term or n^{th} term of a G.P.

- If a and r be the first term and common ratio of G.P. respectively, then its general term is, $a_n = ar^{n-1}$.
- If l is the last term of a G.P. consisting of n terms, then $l = ar^{n-1}$
- If $a, ar, ar^2, ...$ is a finite G.P. consisting of m terms, then the n^{th} term from the end
  $= (m - n + 1)^{th}$ term from beginning
  $= ar^{m-n+1-1} = ar^{m-n}$
- If $a, ar, ar^2, ...$ is a finite G.P. with last term l, then the n^{th} term from the end $= l\left(\frac{1}{r}\right)^{n-1}$.
- Common ratio of a G.P. $= \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$, where
  n = number of terms, a = first term and b = last term of the G.P.

Sum of *n* terms of a G.P.

For Finite G.P. : If S_n denotes the sum of first n terms of G.P. with first term a and common ratio 'r'. Then,

$$S_n = \begin{cases} \frac{a(r^n - 1)}{r-1}, r > 1 \\ \frac{a(1-r^n)}{(1-r)}, r < 1 \end{cases}$$

If $r = 1$ then, $S_n = a + a + a + ... + a$ (n times) $= na$

For Infinite G.P. : $S_n = \frac{a}{1-r}, |r| < 1$

Remark :
If the product of the numbers is given, then in a G.P.,

(i) three numbers are taken as $\frac{a}{r}, a, ar$.

(ii) four numbers are taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

(iii) five numbers are taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.

Properties of Geometric Progression

(i) Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$.
(ii) The reciprocals of the terms of a G.P. are also in G.P.
(iii) If each term of a G.P. be multiplied or divided by a non-zero number, then the sequence obtained is also a G.P.
(iv) If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.

Geometric Mean

(i) G.M. between two numbers
Let a and b be any two positive numbers, then G.M. $= \sqrt{ab}$.

(ii) *n* G.M.'s between two numbers
Let a and b be any two positive numbers, and $G_1, G_2,..., G_n$ be n G.M.'s between them, then

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\therefore \quad G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, ..., G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

$\therefore$ Product of n G.M's $= G_1 G_2 ... G_n = \left(\sqrt{ab}\right)^n$

Relationship between A.M. and G.M.

Let A and G be respectively A.M. and G.M. of two positive real numbers a and b, then

$$A = \frac{a+b}{2} \quad \text{and} \quad G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0 \text{ i.e. } A \geq G$$

Sum to *n* terms of special series

(i) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Evaluate $7^{1/2} \times 7^{1/4} \times 7^{1/8} \times \ldots$ to infinite terms.

2. What is the 15th term defined by the $t_n = \dfrac{n^2}{n+2}$?

3. Divide 69 into three parts which are in A.P. and the product of the two smaller parts is 483.

4. If the fifth term of a G.P. is 81 and second term is 24. Find the common ratio.

5. Find the 15th term of the G.P. $\dfrac{3}{2}, \dfrac{3}{4}, \dfrac{3}{8}, \ldots$

SHORT ANSWER TYPE

6. Find the number of terms of a geometric sequence $\{a_n\}$ if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

7. Find the first five terms of the sequence for which $t_1 = 1$, $t_2 = 2$ and $t_{n+2} = t_n + t_{n+1}$.

8. A man repays a loan of ₹ 3250 by paying ₹ 20 in the first month and then increases the payment by ₹ 15 every month. How long will it take him to clear the loan?

9. If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1 : 3, then find the value of of n.

10. Which term of the sequence $25, 24\frac{1}{4}, 23\frac{1}{2}, 22\frac{3}{4}, \ldots$ is the first negative term?

LONG ANSWER TYPE - I

11. Find the least number of terms of the series $19 + 18\frac{1}{5} + 17\frac{2}{5} + \ldots$ whose sum is negative. Also calculate the exact sum.

12. If $(p + q)^{th}$ term of a G.P. is m and $(p - q)^{th}$ term is n, show that p^{th} term is $\sqrt{mn}$ and 9th term is $m\left(\dfrac{n}{m}\right)^{p/2q}$.

13. Let a, b, c, d, e be five real numbers such that a, b, c are in A.P.; b, c, d are in G.P., $\dfrac{1}{c}, \dfrac{1}{d}, \dfrac{1}{e}$ are in A.P. If $a = 2$ and $e = 18$, find all possible values of b, c and d.

14. Find $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots$ to 10 terms.

15. The arithmetic mean between two positive numbers a and b, where $a > b$, is twice their geometric mean. Prove that $a : b = (2+\sqrt{3}) : (2-\sqrt{3})$.

LONG ANSWER TYPE - II

16. (i) If there are $(2n + 1)$ terms in an A.P., prove that the sum of odd terms and the sum of even terms bear the ratio $(n + 1) : n$.

(ii) If the m^{th} term of an A.P. is $(1/n)$ and its n^{th} term is $(1/m)$, show that the sum of mn terms is $\dfrac{1}{2}(mn + 1)$.

17. Find the sum of the following series to n terms:

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \ldots$$

18. If a^2, b^2, c^2 are in A.P., then prove that the following are also in A.P.

(i) $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ (ii) $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$

19. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

20. (i) If a, b, c, d are in G.P., prove that $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$

(ii) The p^{th}, q^{th} and r^{th} terms of an A.P. as well as those of a G.P. are a, b, c, respectively, prove that $a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$.

SOLUTIONS

1. $7^{1/2} \times 7^{1/4} \times 7^{1/8} \times \ldots$ to infinite terms

$= 7^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots \text{ to } \infty \text{ terms}} = 7^{\frac{1/2}{1-1/2}}$

$[\because \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ is an infinite G.P. with $a = \frac{1}{2}$ and $r = \frac{1}{2}]$

$= 7^1 = 7$

2. Here $t_n = \dfrac{n^2}{n+2}$ $\quad\therefore\quad t_{15} = \dfrac{15^2}{15+2} = \dfrac{225}{17}$

3. Let the three parts of 69 be $a - d, a, a + d$ $(d > 0)$

Then, $(a - d) + a + (a + d) = 69 \Rightarrow 3a = 69 \Rightarrow a = 23$

Given, $a(a - d) = 483$

$\therefore$ $23(23 - d) = 483 \Rightarrow 23 - d = 21 \Rightarrow d = 2$

$\therefore$ The three parts $a - d, a, a + d$ are 21, 23, 25.

4. Here, $a_5 = 81$ and $a_2 = 24$

Let r be the common ratio of G.P.

$\therefore$ $81 = ar^4$ $\quad [\because a_n = ar^{n-1}]$ ...(i)

and $24 = ar$ ...(ii)

Dividing (i) by (ii), we get

$$\frac{ar^4}{ar} = \frac{81}{24} = \frac{27}{8} \Rightarrow r^3 = \left(\frac{3}{2}\right)^3 \Rightarrow r = \frac{3}{2}.$$

5. n^{th} term of a G.P. is given by $a_n = ar^{n-1}$

Here, $a=\frac{3}{2}, r=\frac{1}{2}$ and $n = 15$

$\therefore\ a_{15} = 15^{\text{th}}$ term of the G.P. $=\frac{3}{2}\left(\frac{1}{2}\right)^{15-1}=\frac{3}{2^{15}}$.

6. Let r be the common ratio of G.P., then

$a_n = a_1 r^{n-1} \Rightarrow 96 = 3\cdot r^{n-1} \Rightarrow r^{n-1} = 32$ ...(i)

Now, $S_n=\frac{a(r^n-1)}{r-1}\Rightarrow 189=\frac{3(r^n-1)}{r-1}$

$\Rightarrow 63(r-1) = r^n - 1 \Rightarrow 63r - 62 = r\cdot r^{n-1}$

$\Rightarrow 63r - 62 = r \times 32 \Rightarrow 31r = 62 \Rightarrow r = 2$

Substituting this value of r in (i), we get

$2^{n-1} = 32 = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6.$

Hence, the number of terms = 6.

7. Given, $t_1 = 1, t_2 = 2, t_{n+2} = t_n + t_{n+1}$

Substituting $n = 1$, we get $t_3 = t_1 + t_2 = 1 + 2 = 3$

$n = 2$, we get $t_4 = t_2 + t_3 = 2 + 3 = 5$

$n = 3$, we get $t_5 = t_3 + t_4 = 3 + 5 = 8$

Thus the first five terms of the given sequence are 1, 2, 3, 5, 8.

8. Suppose the loan is cleared in n months. Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

Sum of the amounts = 3250 [Given]

$\Rightarrow \frac{n}{2}\{2\times 20+(n-1)\times 15\}=3250$

$\Rightarrow 3n^2 + 5n - 1300 = 0$

$\Rightarrow (n - 20)(3n + 65) = 0 \Rightarrow n = 20$ $\left[\because\ n\neq\frac{-65}{3}\right]$

Thus, the loan is cleared in 20 months.

9. Let $A_1, A_2, ..., A_n$ be n arithmetic means between 20 and 80 and let d be the common difference of the A.P. $20, A_1, A_2, ..., A_n, 80$. Then, $d=\frac{80-20}{n+1}=\frac{60}{n+1}$

Now, $A_1 = 20 + d \Rightarrow A_1 = 20+\frac{60}{n+1}=20\left(\frac{n+4}{n+1}\right)$

and, $A_n = 20 + nd \Rightarrow A_n = 20+\frac{60n}{n+1}=20\left(\frac{4n+1}{n+1}\right)$

It is given that $\frac{A_1}{A_n}=\frac{1}{3}\Rightarrow\frac{\frac{20(n+4)}{n+1}}{\frac{20(4n+1)}{n+1}}=\frac{1}{3}\Rightarrow\frac{n+4}{4n+1}=\frac{1}{3}$

$\Rightarrow 4n + 1 = 3n + 12 \Rightarrow n = 11$

10. The given sequence $25, 24\frac{1}{4}, 23\frac{1}{2}, 22\frac{3}{4}, ...$ is an A.P. with common difference $d = -\frac{3}{4}$ and first term $a = 25$.

Let n^{th} term of the given A.P. be the first negative term, then $a_n < 0 \Rightarrow 25 + (n-1)\left(-\frac{3}{4}\right) < 0$

$\Rightarrow \frac{103}{4}-\frac{3n}{4}<0 \Rightarrow 103 - 3n < 0 \Rightarrow 103 < 3n$

$\Rightarrow 3n > 103 \Rightarrow n > \frac{103}{3}$ *i.e.*, $n > 34\frac{1}{3}$.

Since 35 is the least natural number satisfying $n > 34\frac{1}{3}$

$\Rightarrow n = 35$

Hence, 35^{th} term of the given sequence is the first negative term.

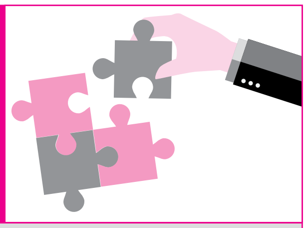
11. The given series $19+18\frac{1}{5}+17\frac{2}{5}+...$ is an arithmetic series with first term a = 19 and common difference $d=-\frac{4}{5}$.

Let the required number of terms be n.

According to given condition sum to n terms < 0

$\Rightarrow \frac{n}{2}\left[2\times 19+(n-1)\left(-\frac{4}{5}\right)\right]<0 \Rightarrow n\left[19-\frac{2}{5}(n-1)\right]<0$.

PUZZLE CORNER



MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics.

In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is multiple of its numbers. For example, if that value is 3 for a two-box cluster, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.

12+		8+		5	10+
	3		4		
12+			9+		
10+		12+	3	1	9+
	8+			7+	
		11+			2

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

$\Rightarrow \frac{n}{5}(97-2n)<0 \Rightarrow -\frac{2}{5}n\left(n-\frac{97}{2}\right)<0$

$\Rightarrow n\left(n-\frac{97}{2}\right)>0 \Rightarrow n<0$ or $n>\frac{97}{2}$.

As n (number of terms) is a positive integer, therefore, the least value of $n = 49$.

Then, sum $=\frac{49}{2}\left[2\times 19+(49-1)\left(-\frac{4}{5}\right)\right] = -\frac{49}{5} = -9\frac{4}{5}$.

12. Let a be the first term and r be the common ratio of the G.P.

Given, $a_{p+q} = m$ and $a_{p-q} = n$

$\Rightarrow ar^{p+q-1} = m$ and $ar^{p-q-1} = n$

$\Rightarrow mn = a^2 r^{p+q-1+p-q-1} = a^2 r^{2p-2}$

$\Rightarrow mn = (ar^{p-1})^2 = (a_p)^2 \Rightarrow a_p = \sqrt{mn}$.

$\Rightarrow p^{th}$ term $= \sqrt{mn}$.

Also, $\frac{m}{n} = \frac{ar^{p+q-1}}{ar^{p-q-1}} = r^{2q}$ ...(i)

$\therefore$ q^{th} term $= ar^{q-1} = \frac{ar^{p+q-1}}{r^p} = \frac{m}{\left(\left(\frac{m}{n}\right)^{\frac{1}{2q}}\right)^p} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

13. Given, a, b, c are in A.P. $\Rightarrow$ $b = \frac{a+c}{2}$ ...(i)

b, c, d are in G.P. $\Rightarrow$ $c^2 = bd$ ...(ii)

and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. $\Rightarrow$ $\frac{2}{d} = \frac{1}{c}+\frac{1}{e} = \frac{e+c}{ce}$

$\Rightarrow d = \frac{2ce}{c+e}$ ...(iii)

From (ii), $c^2 = bd = \frac{a+c}{2}\cdot\frac{2ce}{c+e} = \frac{(a+c)ce}{c+e}$

$\Rightarrow c = \frac{(a+c)e}{c+e}$ $\Rightarrow c^2 + ce = ae + ce \Rightarrow c^2 = ae$ ...(iv)

Given, $a = 2, e = 18$ $\therefore$ From (iv) $c^2 = 36 \Rightarrow c = \pm 6$

From (i), $b = \frac{a+c}{2} = \frac{2\pm 6}{2} = 4, -2$

From (ii), $d = \frac{c^2}{b} = \frac{36}{4}$ or $\frac{36}{-2} = 9$ or -18

Thus, $c = 6, b = 4, d = 9$ or $c = -6, b = -2, d = -18$.

14. Let $S_n = (3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + ...$ to n terms ...(i)

Now, n^{th} term of sequence 3, 5, 7, ...

$= 3 + (n - 1)\,2 = 2n + 1$

and n^{th} term of the series $3^3 + 5^3 + 7^3 + ... = (2n + 1)^3$

n^{th} term of the sequence 2, 4, 6, ... $= 2 + (n - 1)2 = 2n$

$\therefore$ n^{th} term of the series $2^3 + 4^3 + 6^3 + ... = (2n)^3 = 8n^3$

From (i), n^{th} term of the given series

$a_n = (2n + 1)^3 - 8n^3 = 12n^2 + 6n + 1$

$\therefore$ $S_n = \Sigma\, 12n^2 + \Sigma\, 6n + \Sigma\, 1 = 12\,\Sigma n^2 + 6\Sigma n + n$

$= \frac{12n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$

$= n[2(2n^2 + 3n + 1) + 3n + 3 + 1]$

$= n\,(4n^2 + 9n + 6) = 4n^3 + 9n^2 + 6n$

Putting $n = 10$, we get

$S_{10} = 4\cdot 10^3 + 9\cdot 10^2 + 6\cdot 10 = 4960$

15. Let A be the A.M. and G be the G.M. between a and b.

Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Given, $A = 2G \Rightarrow \frac{a+b}{2} = 2\sqrt{ab} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$

$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3}{1}$ [By componendo and dividendo]

$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{3}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$

$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ [By componendo and dividendo]

$\Rightarrow \frac{a}{b} = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 = \frac{4+2\sqrt{3}}{4-2\sqrt{3}} \Rightarrow \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$.

16. (i) Let a be the first term and d be the common difference of the given A.P.

Then, $a_k = a + (k - 1)d$ ...(i)

Let S_1 & S_2 denote the sum of all odd terms and the sum of all even terms respectively, then

$\therefore$ $S_1 = a_1 + a_3 + a_5 + ... + a_{2n+1} = \frac{(n+1)}{2}\{a_1 + a_{2n+1}\}$

$= \frac{(n+1)}{2}\{a + a + (2n + 1 - 1)d\} = (n + 1)(a + nd)$

and $S_2 = a_2 + a_4 + a_6 + ... + a_{2n} = \frac{n}{2}[a_2 + a_{2n}]$

$= \frac{n}{2}[(a + d) + (a + (2n - 1)d)] = n\,(a + nd)$

$\therefore \frac{S_1}{S_2} = \frac{(n+1)(a+nd)}{n(a+nd)} = \frac{n+1}{n}$

Hence, the required ratio is $(n + 1) : n$.

(ii) Let a be the first term and d be the common difference of the given A.P.

Then, $a_m = \frac{1}{n}$ and $a_n = \frac{1}{m}$ [Given]

Now, $a_m = \frac{1}{n} \Rightarrow a + (m - 1)d = \frac{1}{n}$ ...(i) and $a_n = \frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m}$ ...(ii)

On subtracting (ii) from (i), we get

$(m-n)d = \left(\frac{1}{n} - \frac{1}{m}\right) = \frac{(m-n)}{mn} \Rightarrow d = \frac{1}{mn}$

Putting $d = \frac{1}{mn}$ in (i), we get

$a + \frac{(m-1)}{mn} = \frac{1}{n} \Rightarrow a = \left\{\frac{1}{n} - \frac{(m-1)}{mn}\right\} = \frac{1}{mn}$

Thus, $a = \frac{1}{mn}$ and $d = \frac{1}{mn}$

$\therefore\ S_{mn} = \frac{mn}{2}\{2a + (mn-1)d\}$

$= \frac{mn}{2}\left\{\frac{2}{mn} + \frac{(mn-1)}{mn}\right\}$ $\left[\because a = \frac{1}{mn} \text{ and } d = \frac{1}{mn}\right]$

$= \frac{1}{2}(mn+1)$

Hence, the sum of mn terms is $\frac{1}{2}(mn+1)$.

17. Let a_n be the n^{th} term and S_n be the sum to n terms of the given series.

$$\therefore\ a_n = \frac{1}{[(n^{th} \text{ terms of seq. } 1, 2, 3, \ldots) \times (n^{th} \text{ term of the seq. } 2,3,4,\ldots) \times (n^{th} \text{ term of the seq. } 3,4,5,\ldots)]}$$

$= \frac{1}{n(n+1)(n+2)}$

Let $a_n = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$ ...(i)

Then, A = value of $\frac{1}{(n+1)(n+2)}$ when $n = 0$

$\therefore\ A = \frac{1}{(0+1)(0+2)} = \frac{1}{2}$

B = value of $\frac{1}{n(n+2)}$ when $n + 1 = 0$ *i.e.*, $n = -1$

$\therefore\ B = \frac{1}{-1(-1+2)} = -1$

C = value of $\frac{1}{n(n+1)}$ when $n + 2 = 0$ *i.e.*, $n = -2$

$\therefore\ C = \frac{1}{-2(-2+1)} = \frac{1}{2}$

From (i), $a_n = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$

Putting $n = 1, 2, 3, \ldots, n$, we get

$a_1 = \frac{1}{2\cdot1} - \frac{1}{2} + \frac{1}{2\cdot3}$,

$a_2 = \frac{1}{2\cdot2} - \frac{1}{3} + \frac{1}{2\cdot4}$

$a_3 = \frac{1}{2\cdot3} - \frac{1}{4} + \frac{1}{2\cdot5}$

... ...

... ...

$a_{n-2} = \frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2n}$

$a_{n-1} = \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}$

$a_n = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$

Adding $a_1, a_2, a_3, \ldots, a_n$ we get,

$S_n = \frac{1}{2\cdot1} - \frac{1}{2} + \frac{1}{2\cdot2} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$

$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

18. (i) It is given that a^2, b^2, c^2 are in A.P.

$\therefore\ b^2 - a^2 = c^2 - b^2$

$\Rightarrow (b-a)(b+a) = (c-b)(c+b) \Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b}$

$\Rightarrow \frac{(b+c)-(a+c)}{b+c} = \frac{(c+a)-(b+a)}{a+b}$

$\Rightarrow \frac{(b+c)-(a+c)}{(a+c)(b+c)} = \frac{(c+a)-(b+a)}{(a+b)(a+c)}$

$\left[\text{Multiplying both side by } \frac{1}{a+c}\right]$

$\Rightarrow \frac{1}{a+c} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{a+c}$

$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

(ii) It is given that a^2, b^2, c^2 are in A.P., then from (i) part

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow \frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$ are also in A.P.

$\Rightarrow 1 + \frac{a}{b+c}, 1 + \frac{b}{c+a}, 1 + \frac{c}{a+b}$ are in A.P.

$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

19. Let the numbers in G.P. be a, ar, ar^2. It is given that the sum of these numbers is 56.

$\therefore\ a + ar + ar^2 = 56 \Rightarrow a + ar^2 = 56 - ar$ ...(i)

It is also given that $a - 1$, $ar - 7$ and $ar^2 - 21$ are in A.P.

$\therefore\ \ 2(ar-7) = (a-1) + (ar^2-21) \Rightarrow 2ar = a + ar^2 - 8$

$\Rightarrow\ a + ar^2 = 2ar + 8$ ...(ii)

From (i) and (ii), we have

$2ar + 8 = 56 - ar \Rightarrow ar = 16 \Rightarrow r = \frac{16}{a}$ ...(iii)

Putting $r = \frac{16}{a}$ in (i), we get

$a + 16 + \frac{256}{a} = 56 \Rightarrow a^2 - 40a + 256 = 0$

$\Rightarrow\ (a-32)(a-8) = 0 \Rightarrow a = 8, 32$

Putting $a = 8$ in $r = \frac{16}{a}$, we get $r = \frac{16}{8} = 2$

Putting $a = 32$ in $r = \frac{16}{a}$, we get $r = \frac{16}{32} = \frac{1}{2}$

When $a = 8$ and $r = 2$, we obtain 8, 16 and 32 as the numbers of G.P.

When $a = 32$ and $r = \frac{1}{2}$, we obtain 32, 16, 8 as the numbers of G.P.

Hence, the numbers are 8, 16 and 32 or 32, 16 and 8.

20. (i) Let r be the common ratio of the G.P. a, b, c, d.

Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

$\therefore$ L.H.S. $= (a^2+b^2+c^2)(b^2+c^2+d^2)$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$= a^4r^2(1+r^2+r^4)^2$ ...(i)

And, R.H.S. $= (ab+bc+cd)^2$

$= (a^2r + a^2r^3 + a^2r^5)^2 = a^4r^2(1+r^2+r^4)^2$ ...(ii)

Hence, from (i) and (ii)

$(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$.

(ii) Let x be the first term and d be the common difference of an A.P. Then,

$x + (p-1)d = a$ ...(i) $x + (q-1)d = b$ ...(ii)

$x + (r-1)d = c$ ...(iii)

On subtracting (ii) from (i), we get

$a - b = (p-q)d$ ...(iv)

On subtracting (iii) from (ii), we get

and $b - c = (q-r)d$ ...(v)

Now, let A be the first term and R be the common ratio of the G.P. Then,

$AR^{p-1} = a$ ...(vi) $AR^{q-1} = b$ ...(vii) and $AR^{r-1} = c$ ...(viii)

On dividing (vi) by (vii) and (vii) by (viii), we get

$\Rightarrow\ \frac{a}{b} = R^{p-q}$ and $\frac{b}{c} = R^{q-r} \Rightarrow \left(\frac{a}{b}\right)^{\frac{1}{p-q}} = \left(\frac{b}{c}\right)^{\frac{1}{q-r}} = R$

$\Rightarrow\ \left(\frac{a}{b}\right)^{\frac{d}{a-b}} = \left(\frac{b}{c}\right)^{\frac{d}{b-c}}$ [Using (iv) and (v)]

$\Rightarrow\ \left(\frac{a}{b}\right)^{b-c} = \left(\frac{b}{c}\right)^{a-b} \Rightarrow \frac{a^{b-c}}{b^{b-c}} = \frac{b^{a-b}}{c^{a-b}}$

$\Rightarrow\ a^{b-c} \cdot c^{a-b} = b^{a-b+b-c} = b^{-c+a}$

$\Rightarrow\ a^{b-c} \cdot c^{a-b} = \frac{1}{b^{c-a}} \Rightarrow a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1.$

◈◈

MPP-6 MONTHLY Practice Problems

Contd. from page no. 63

	P	Q	R	S
(a)	1, 2	3, 4	1	1, 2
(b)	3, 4	2	1	1, 3
(c)	1	2	3	4
(d)	1, 2	1, 3	3	4

Integer Answer Type

17. If $\int \frac{2e^{5x} + e^{4x} - 4e^{3x} + 4e^{2x} + 2e^x}{(e^{2x}+4)(e^{2x}-1)^2}dx = \tan^{-1}\left(\frac{e^x}{2}\right)$

$-\frac{K}{248(e^{2x}-1)} + C$, then K is equal to

18. If $I = \int \frac{dx}{1+\sqrt{x^2+2x+2}} = \frac{A}{7}$

$\log\left|x+1+\sqrt{x^2+2x+2}\right| - \frac{\sqrt{x^2+2x+2}-1}{x+1} + C$

then A is equal to

19. If $f(x) = \sqrt{x}$, $g(x) = e^x - 1$, and $\int fog(x)dx =$

$A\,fog(x) + B\tan^{-1}(fog(x)) + C$, then $A + B$ is equal to

20. Let $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$ and $g(0) = 0$, then the value of $8g(\pi/2)$ is

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SELF CHECK

No. of questions attempted ......

No. of questions correct ......

Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MPP-6 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Sequences and Series

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If three positive real numbers a, b, c are in A.P. such that $abc = 4$, then the minimum value of b is
(a) $2^{1/3}$ (b) $2^{2/3}$
(c) $2^{1/2}$ (d) $2^{3/2}$

2. If a, x, b are in A.P., a, y, b are in G.P. and a, z, b are in H.P. such that $x = 9z$ and $a > 0$, $b > 0$, then
(a) $|y| = 3z$ and $x = 3|y|$
(b) $y = 3|z|$ and $|x| = 3y$
(c) $2y = x + z$
(d) None of these

3. If a, b, c, are in A.P., then $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ will be in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

4. If $(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{128}) = \sum_{r=0}^{n} x^r$, then '$n$' is equal to
(a) 256 (b) 255
(c) 254 (d) None of these

5. If the first, fifth and last terms of an A.P. is l, m, p, respectively, and sum of the A.P. is $\frac{(l+p)(4p+m-5l)}{k(m-l)}$, then k is
(a) 2 (b) 3 (c) 4 (d) 5

6. If a is the A.M. of b and c and two geometric means are G_1 and G_2 are inserted between b and c such that $G_1^3 + G_2^3 = \lambda\, abc$, then $\lambda =$
(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) 3

One or More Than One Option(s) Correct Type

7. If a, b, c are in H.P., then which of the following is/are true?
(a) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.
(b) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$
(c) $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in G.P.
(d) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

8. Sum to n terms of the series
$S = 1^2 + 2(2)^2 + 3^2 + 2(4)^2 + 5^2 + 2(6^2) + \ldots$ is
(a) $1/2n\,(n + 1)^2$, when n is even
(b) $1/2n^2\,(n + 1)$, when n is odd
(c) $1/4n^2\,(n + 2)$, when n is odd
(d) $1/4n(n + 2)^2$, when n is even.

9. $\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \ldots$ upto n terms is equal to

(a) $\frac{\sqrt{3n+2}-\sqrt{2}}{3}$ (b) $\frac{n}{\sqrt{3n+2}+\sqrt{2}}$

(c) less than n (d) less than $\sqrt{\frac{n}{3}}$

10. Given that $x + y + z = 15$ when a, x, y, z, b are in A.P. and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$ when a, x, y, z, b are in H.P. Then

(a) G.M. of a and b is 3
(b) one possible value of $a + 2b$ is 11
(c) A.M. of a and b is 6
(d) greatest value of $a - b$ is 8

11. If d, e, f are in G.P. and the two quadratic equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then

(a) $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in H.P. (b) $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in G.P.

(c) $2dbf = aef + cde$ (d) $b^2df = ace^2$

12. The p^{th} term (T_p) of H.P. is $q(p + q)$ and q^{th} term T_q is $p(p + q)$ when $p > 1, q > 1$, then
(a) $T_{p+q} = pq$ (b) $T_{pq} = p + q$
(c) $T_{p+q} > T_{pq}$ (d) $T_{pq} > T_{p+q}$

13. If $\frac{1}{a}+\frac{1}{c}=\frac{1}{2b-a}+\frac{1}{2b-c}$, then

(a) a, b, c are in A.P. (b) $a, \frac{b}{2}, c$ are in A.P.

(c) $a, \frac{b}{2}, c$ are in H.P. (d) $a, 2b, c$ are in H.P.

Comprehension Type

Let $a_1, a_2, \ldots$ be an A.P. and $b_1, b_2, \ldots$ be a G.P. The sequence $c_1, c_2, c_3, \ldots$ is such that $c_n = a_n + b_n\ \forall\ n \in N$ and $c_1 = 1, c_2 = 4, c_3 = 15, c_4 = 2$.

14. The common ratio of G.P. is
(a) -2 (b) -3 (c) 2 (d) 3

15. The common difference of A.P. is
(a) 3 (b) 4 (c) 5 (d) 6

Matrix Match Type

16. Let $A_1, A_2, \ldots, A_n$ and $G_1, G_2, \ldots., G_n$ be n A.M.'s and n G.M.'s respectively between x and 1 (where $x \in (0,1)$) such that $\sum_{r=1}^{\infty} x^r = \frac{1}{3}$ and $\sum_{r=1}^{n} A_r = 5$. Then

	Column-I		Column-II
P.	$\sum_{r=1}^{n} r^2$ is divisible by	1.	9
Q.	n is divisible by	2.	8
R.	$\left(\frac{G_n}{G_2}\right)^3$ is divisible by integer	3.	6
S.	Value of $1/x$ is	4.	4

	P	Q	R	S
(a)	3, 2	4, 2	2	2
(b)	2, 4	3	4	1, 4
(c)	3, 4	2, 4	2, 4	4
(d)	3, 4	1, 4	2	4

Numerical Answer Type

17. If the sum of an infinite G.P. is equal to the maximum value of $f(x) = x^3 + 2x - 8$ in the interval $[-1, 4]$ and the sum of first two terms of the G.P. is 8. Then, the common ratio of the G.P. is

18. If nine arithmetic means and nine harmonic means are inserted between 2 and 3 alternatively, then the value of $A+\frac{6}{H}$ (where A is any of the A.M.'s and H the corresponding H.M.) is

19. Consider the 10 numbers $ar, ar^2, ar^3, \ldots ar^{10}$. Their sum is 18 and the sum of their reciprocals is 6. If the product of these 10 numbers is a^b (where a and b are prime), then the value of $a + b$ is

20. Let $a_1, a_2, a_3, \ldots, a_{201}$ are in G.P. with $a_{101} = 25$ and $\sum_{i=1}^{201} a_i = 625$, then the value of $\sum_{i=1}^{201} \frac{1}{a_i}$ equals to

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